Introduction to Compartmental Models

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University of California, Berkeley

Adapted from slides by:

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Goals for this lecture

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- Understand the difference between statistical and mechanistic models
  Comprendre la différence entre les modèles statistiques et mécanistes.

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- Example: population growth, predator prey, SIR models

Compartmental/Mechanistic/Mathematical Models

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   Les compartiments et les taux de transition sont déterminés par les systèmes biologiques

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4. Rates of transferring between compartments are expressed mathematically
   Taux de transition entre les compartiments sont exprimés mathématiquement

How are these different from statistical models?

En quoi sont-ils différents des modèles statistiques?

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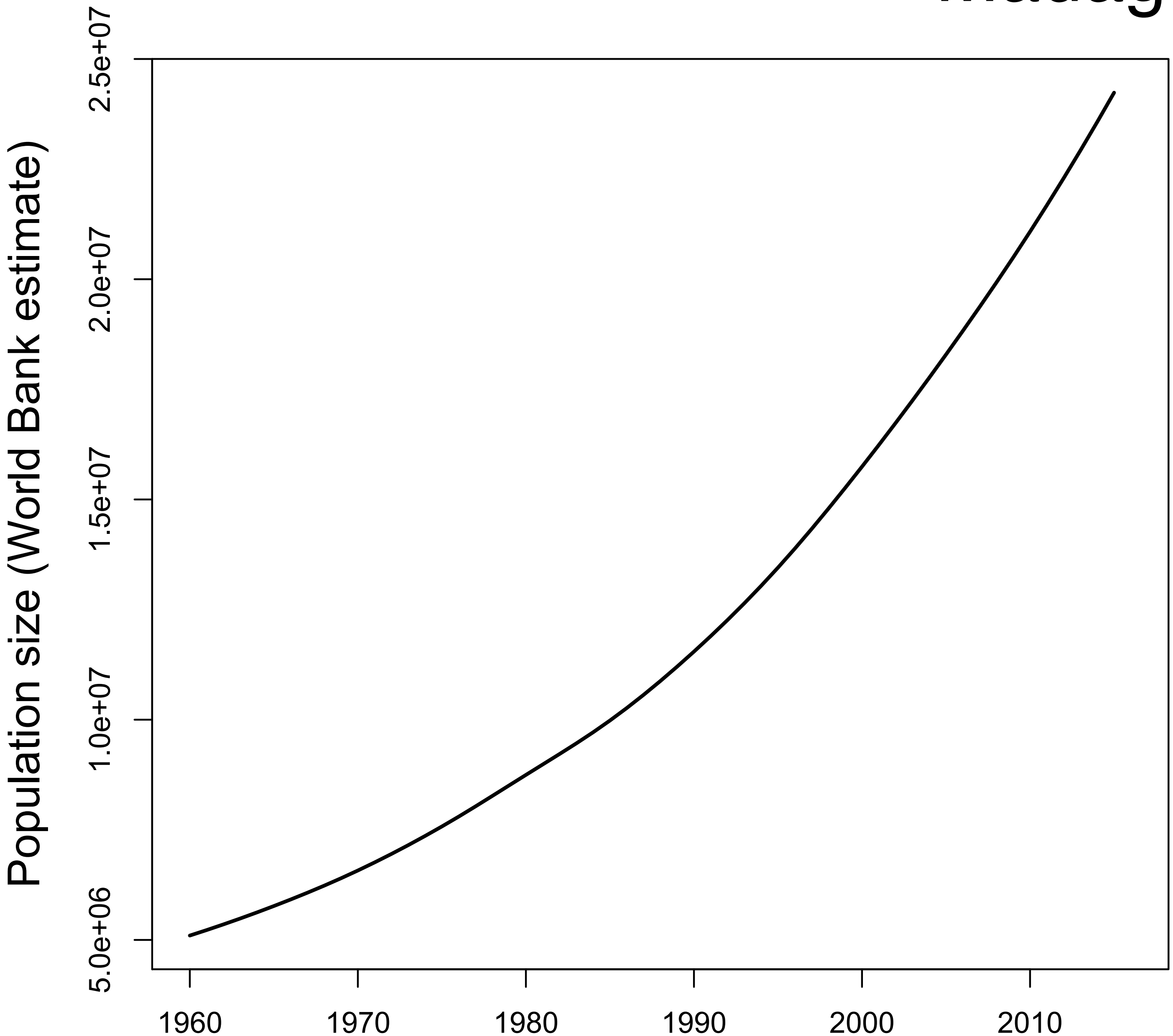
En quoi sont-ils différents des modèles statistiques?

Make explicit hypotheses about biological mechanisms that drive infection dynamics (may not be realistic, but still explicit)

Faire des hypothèses explicites sur les mécanismes biologiques qui régissent la dynamique de l'infection (peut ne pas être réaliste, mais toujours explicite)

1. Simple Population Models
1. Modèles simples de population

Madagascar



http://databank.worldbank.org

The basic population model

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Compartmental models (Mechanistic Models)

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4. Individuals within a compartment are homogenously mix

How does the population of Madagascar grow over time?

Comment est-ce que la population de Madagascar s'augmente avec le passage du temps?

The basic population model

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Madagascar (N)

How does the population grow?

The basic population model

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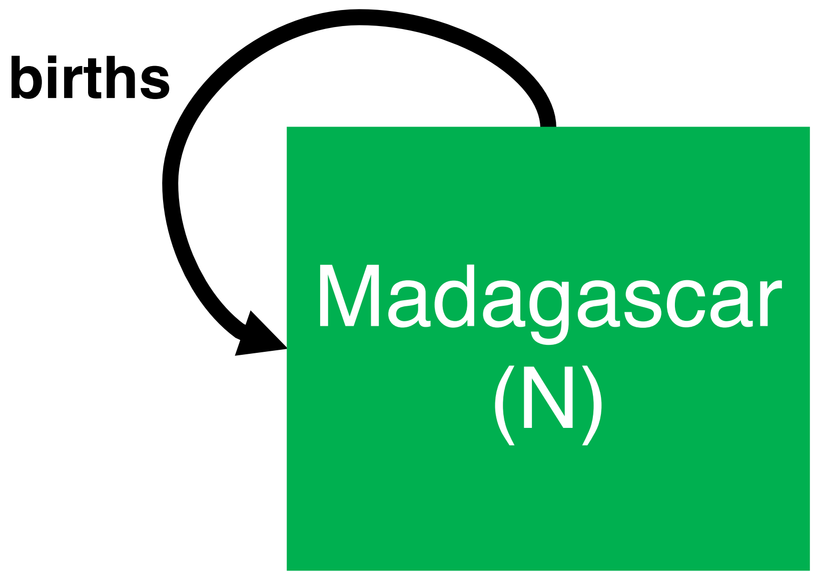

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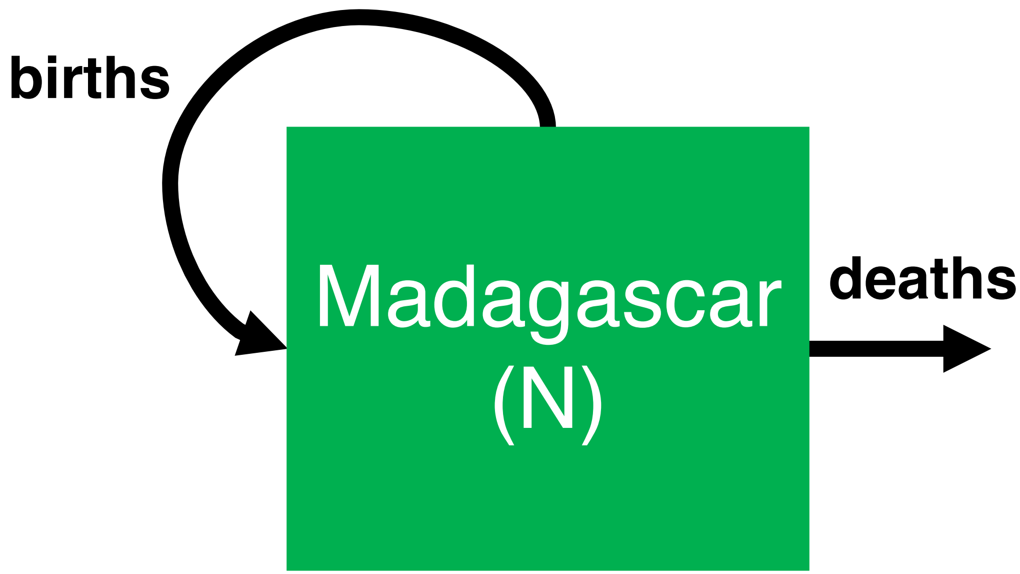

How does the population decrease?

The basic population model

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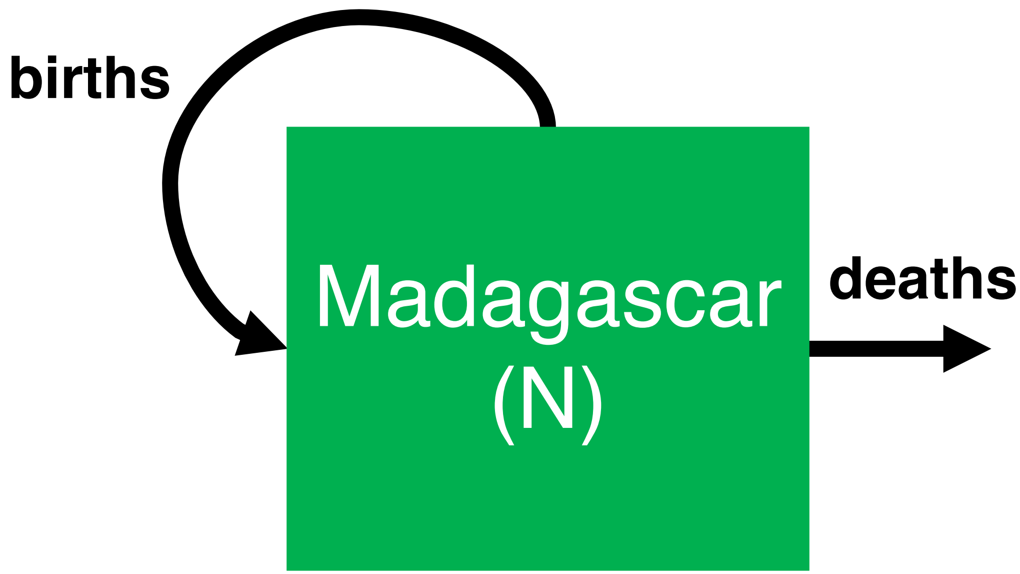

$N_{t+1} =$

The basic population model

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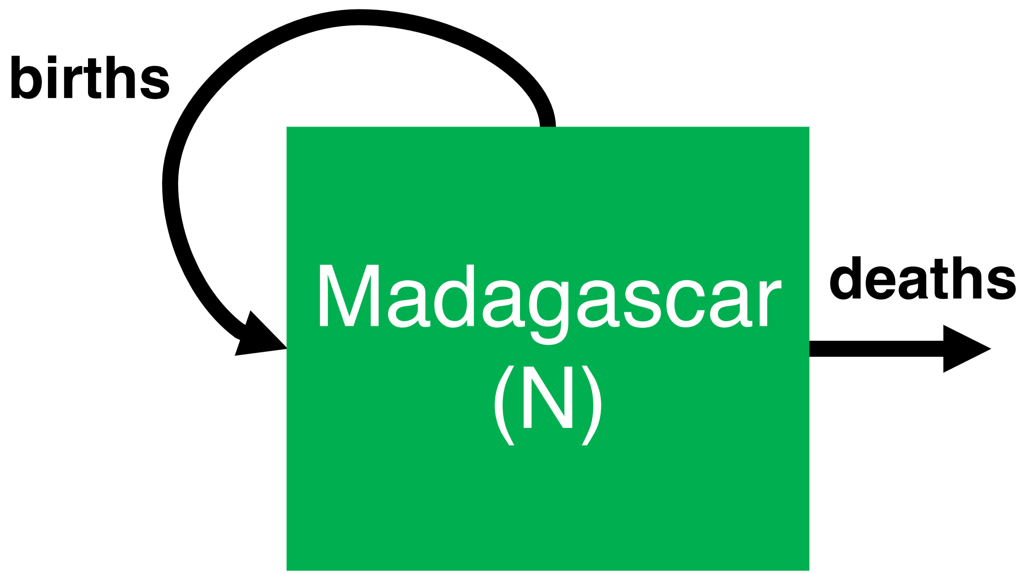

$$N_{t+1}=births*N_t$$

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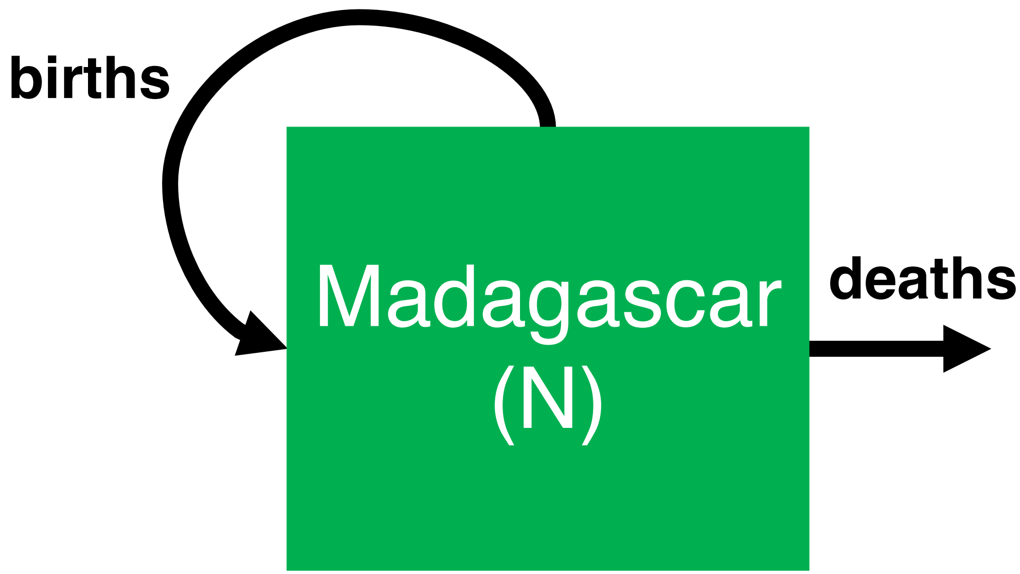

$$N_{t+1}=births*N_t-deaths*N_t$$

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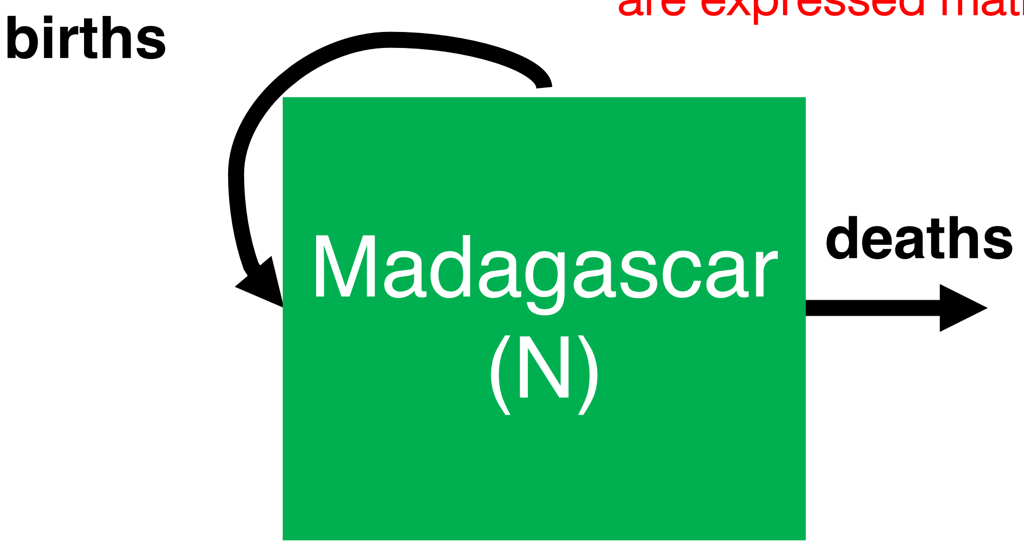


$$N_{t+1} = births * N_t - deaths * N_t$$

$$N_{t+1} = (births - deaths) * N_t$$

$$N_{t+1} = \lambda * N_t$$

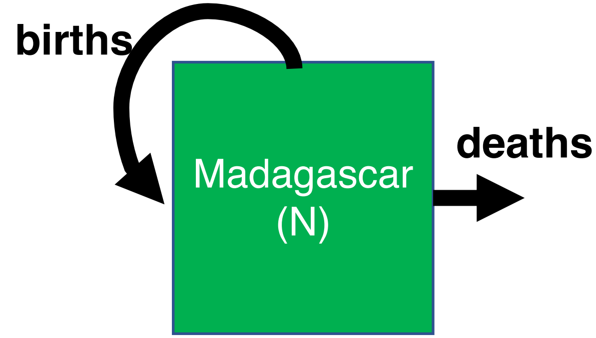
λ=pop intrinsic growth rate

The basic population model

$$\lambda = N_{t+1}/N_t$$

pop size at t

pop size at t+1

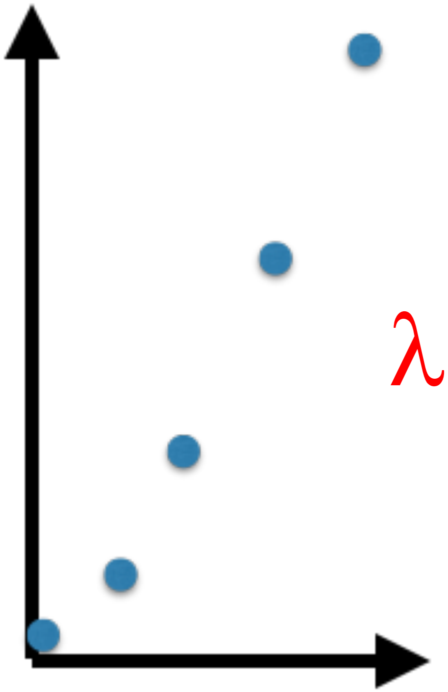


Population rate of increase

Taux d'accroissement de la population

The basic population model

Discrete time



$$\lambda = N_{t+1}/N_t$$

$N_1 = \lambda N_0$
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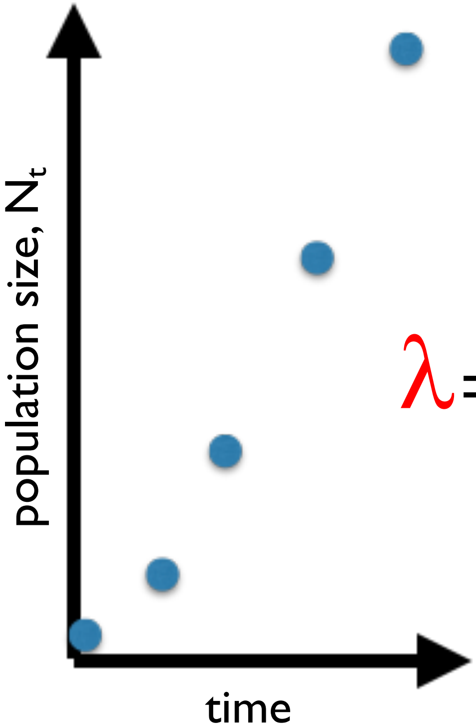
The basic population model

Discrete time



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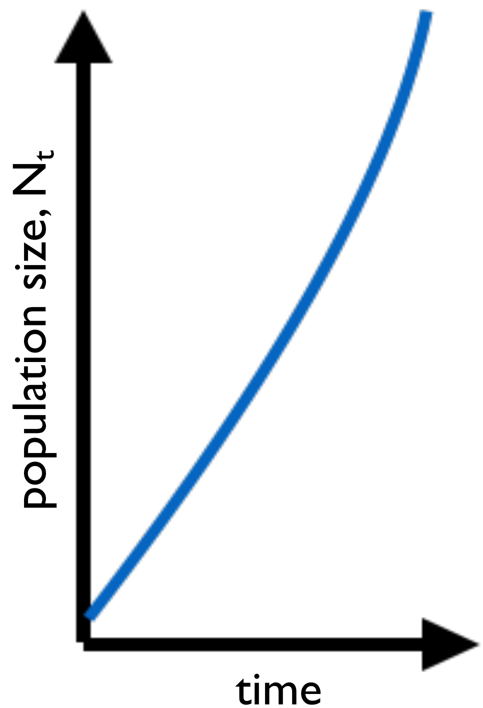
Continuous time

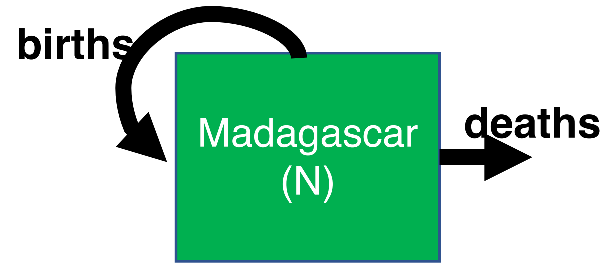


$r = [N(t)-N(0)]/t$

!as $t \to 0$

$dN/dt = rN$

The basic population model



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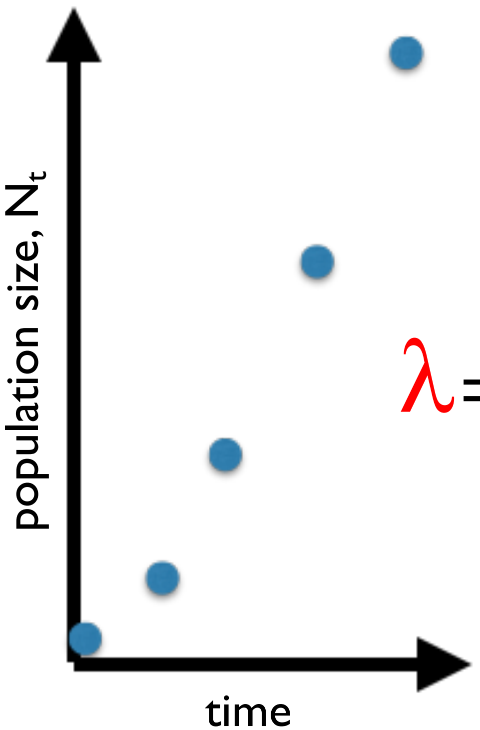

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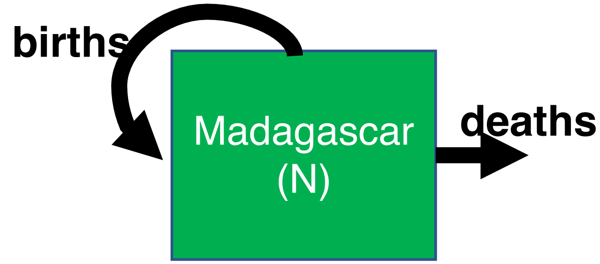
Continuous time

$dN(t)/dt = rN(t)$

The basic population model



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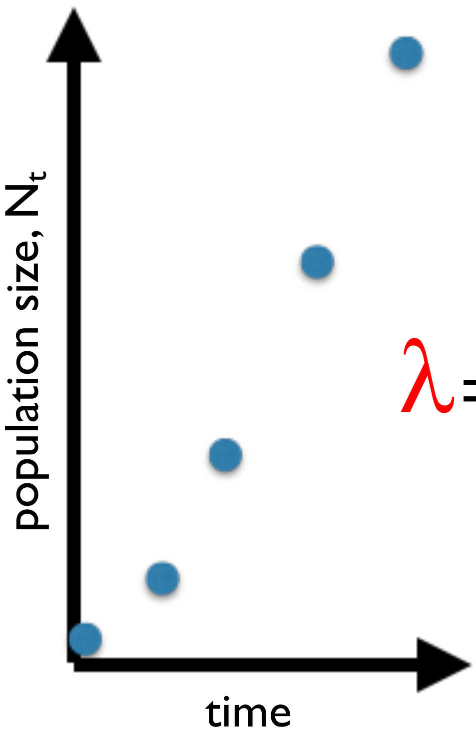

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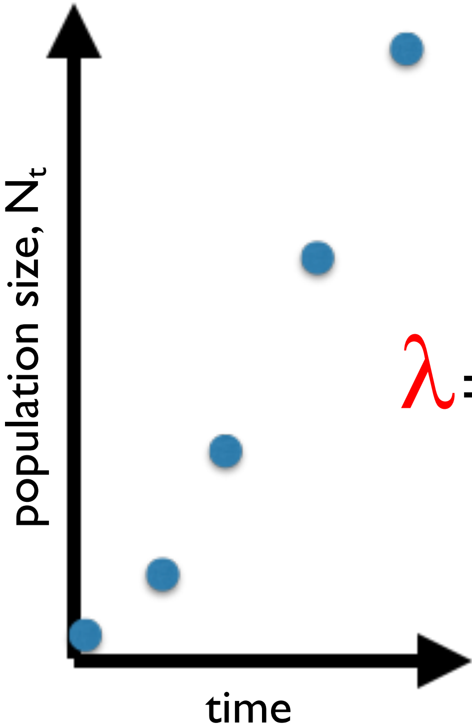
$dN(t)/dt = rN(t)$

Separation of variables:

$dN(t)/N(t) = r\, dt$

The basic population model

Discrete time



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Continuous time

$dN(t)/dt = rN(t)$

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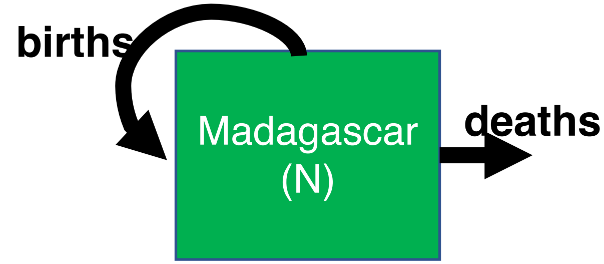
Integrate both sides:
$\int dN(t)/N(t) = \int r\,dt$

The basic population model



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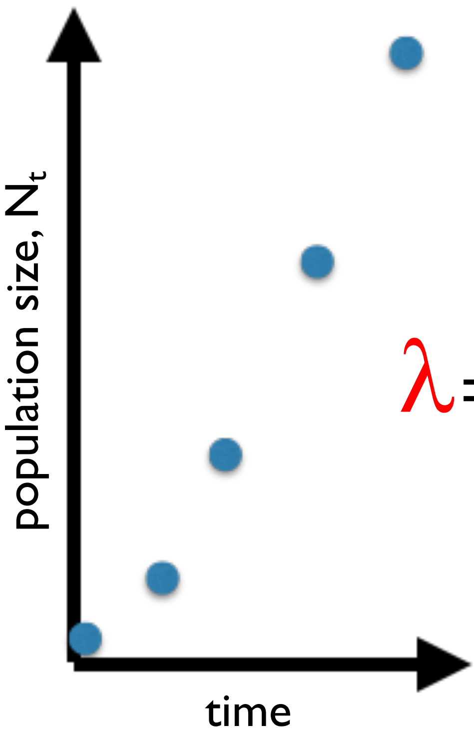

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Separation of variables:

$dN(t)/N(t) = r\ dt$

Integrate both sides:

$\int dN(t)/N(t) = \int r\ dt$

By definition:

$\log(N(t)) = rt + c$

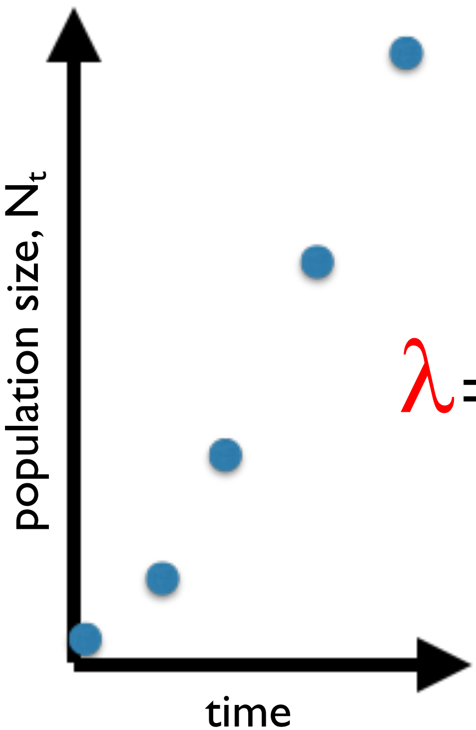
Take exponentials:

$N(t) = e^{rt+c} = Ce^{rt}$

$N(t) = N(0)e^{rt}$

The basic population model

Discrete time



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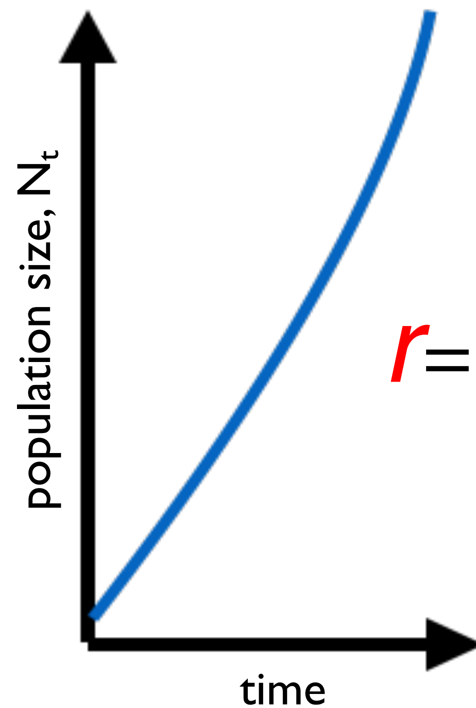


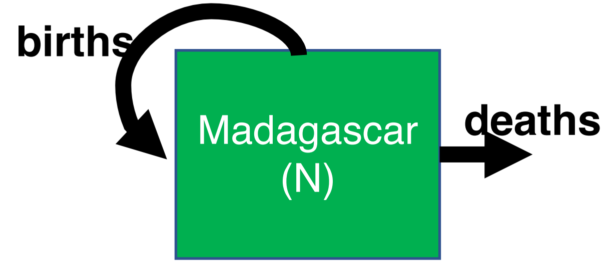
$r = N(t) - N(0)/t$

!as $t \rightarrow 0$

$dN(t)/dt = rN(t)$

$N(t) = N(0)e^{rt}$

The basic population model



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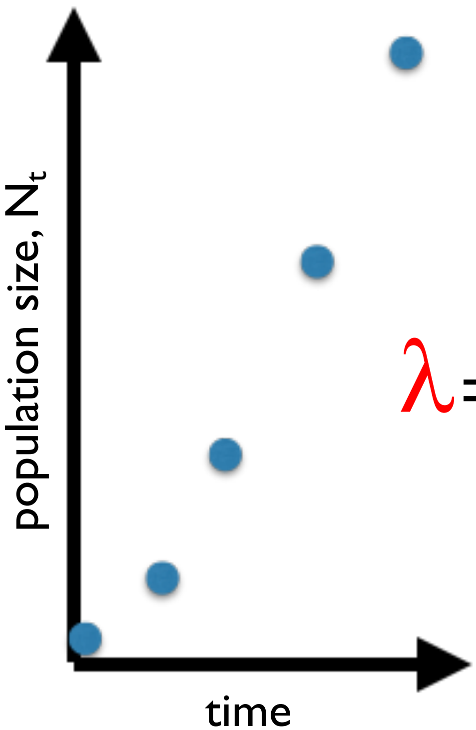

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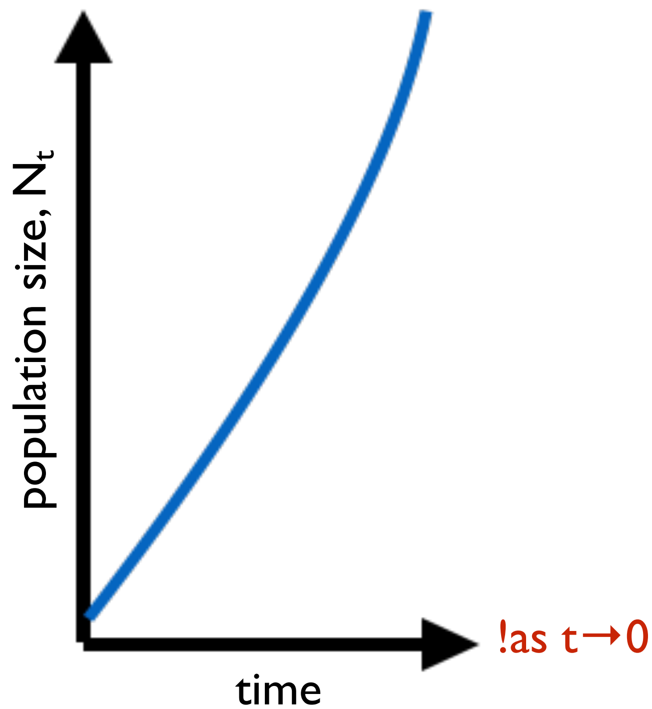
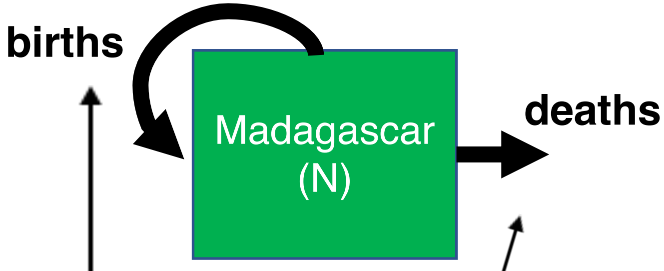
Continuous time



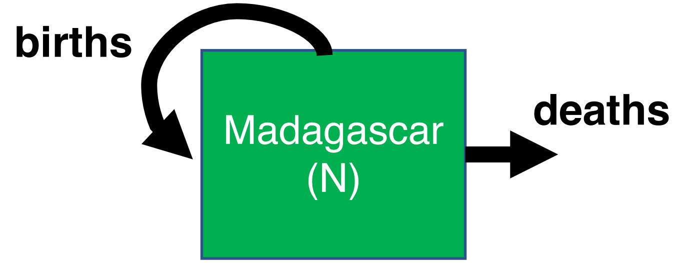
Continuous models can be discretized; discrete models can be approximated by continuous ones. The appropriate framing may depend on the data / question.

The basic population model

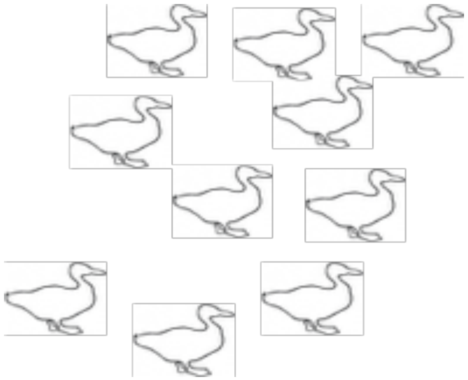


What about those rates?
Are they the same every year?
And in every person?

The basic population model



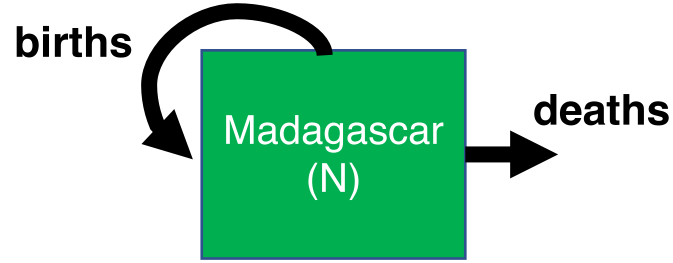
starting population



probability of death = 0.5

if deterministic “always the same”

The basic population model



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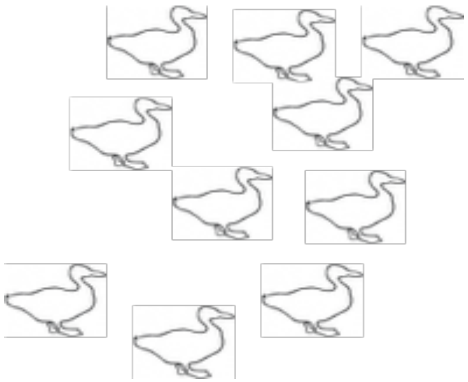

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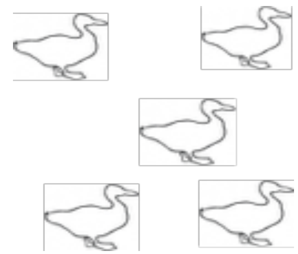

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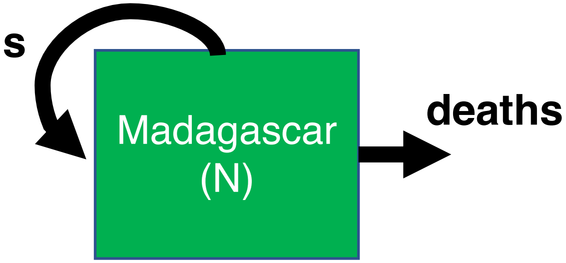

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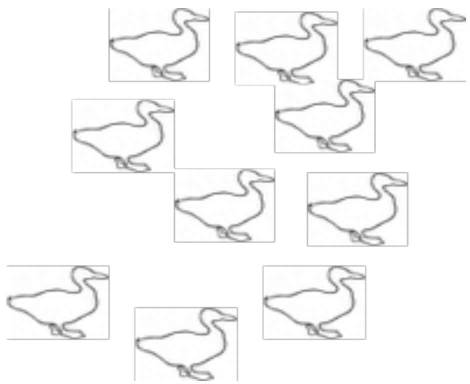

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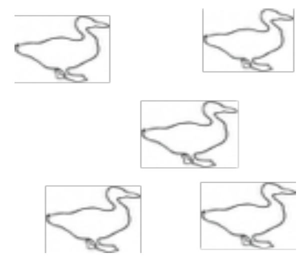

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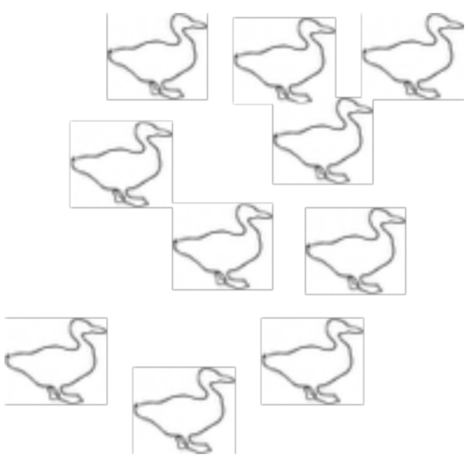

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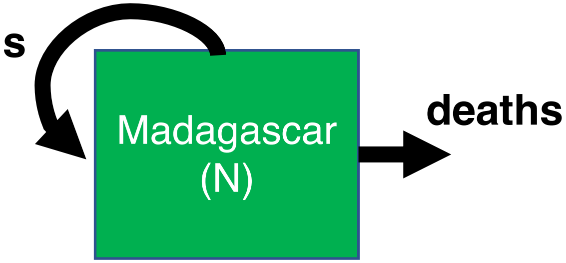
if stochastic? “up to chance”

The basic population model



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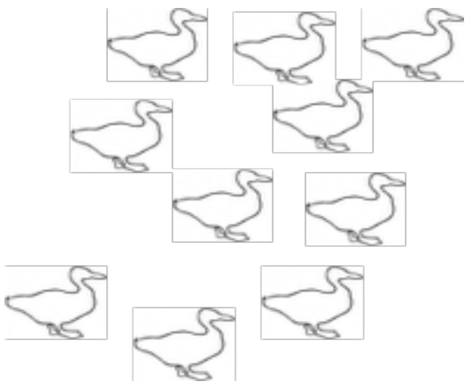

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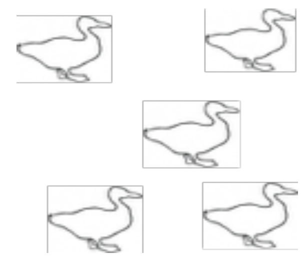

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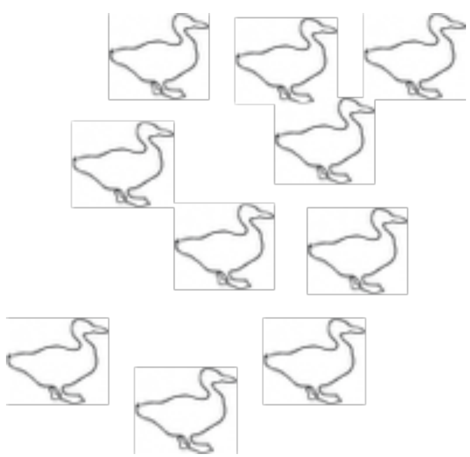

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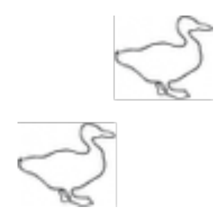
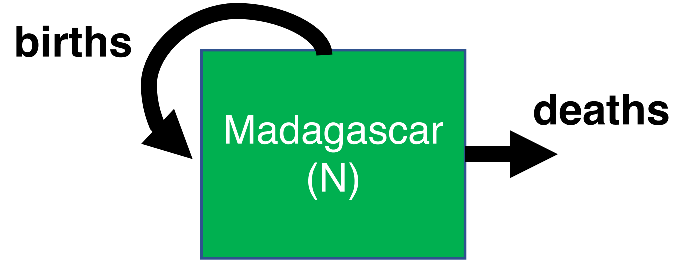


The basic population model



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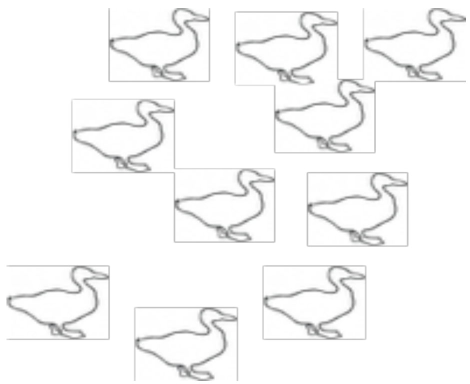

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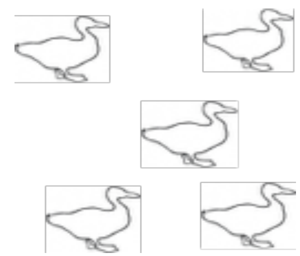

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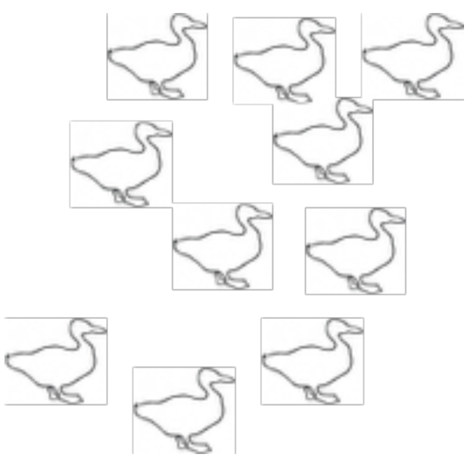

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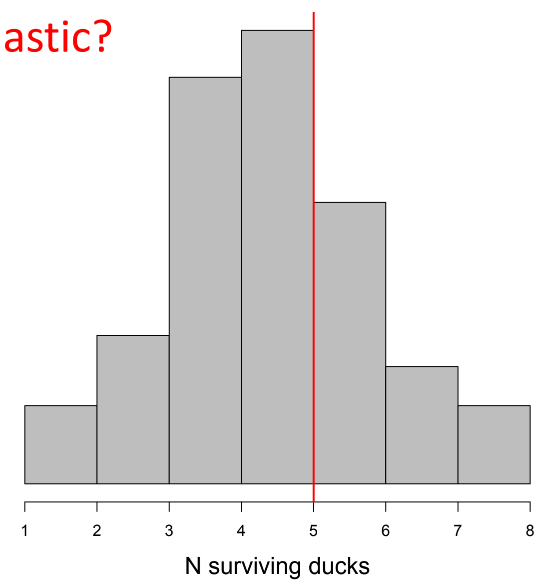
if stochastic?

rbinom(200,10,0.5)



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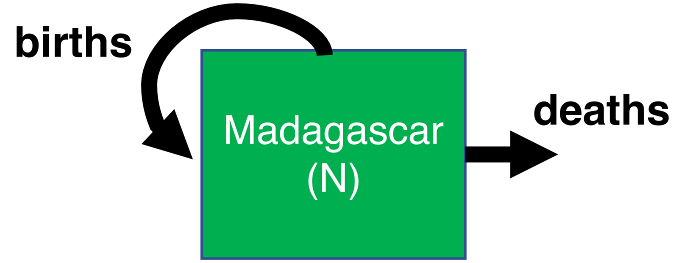
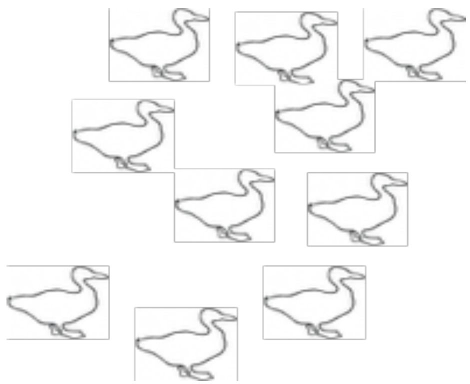
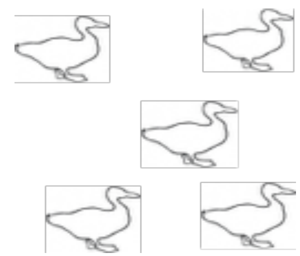


starting population



probability of death = 0.5

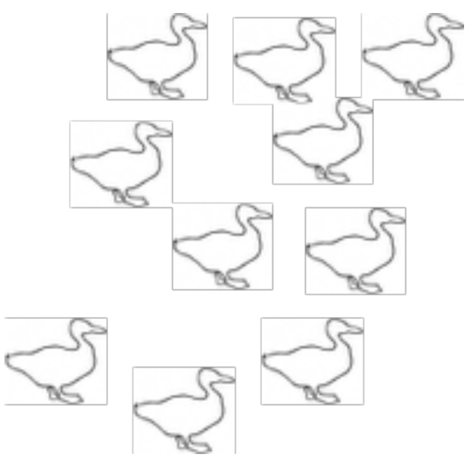
if deterministic "always the same"



If you test your 10 ducks many times, on average you get 5

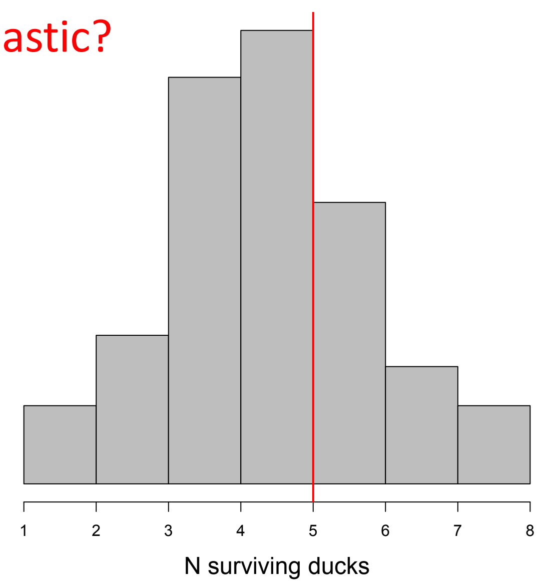
rbinom(200,10,0.5)

starting population



probability of death = 0.5

if stochastic?



Stochasticity matters for ***statistical design***, and ***projecting future population growth***….

It has been suggested that it might also have been a key element in the ***evolution of the unique fauna and flora of Madagascar.***

Evolution in the hypervariable environment of Madagascar

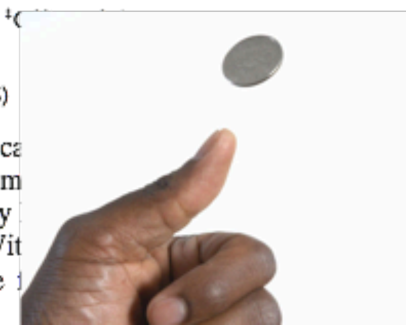
Robert E. Dewar*† and Alison F. Richard‡

*McDonald Institute of Archaeological Research, University of Cambridge, Downing Street, Cambridge CB2 3ER, England; and ‡
Vice-Chancellor, University of Cambridge, Cambridge CB2 1TN, England

Communicated by Henry T. Wright, University of Michigan, Ann Arbor, MI, June 29, 2007 (received for review August 26, 2005)

We show that the diverse ecoregions of Madagascar share one distinctive climatic feature: unpredictable intra- or interannual precipitation compared with other regions with comparable rainfall. Climatic unpredictability is associated with unpredictable patterns of fruiting and flowering. It is argued that these features

are primates; however, in Madagasca
medium- to large-sized frugivorous lem
of fruit in the diets of extant Malagasy
with other primate communities (8). Wit
primate species tend to include more



Key concepts

- Compartmental/mechanistic/mathematical models

  Modèles en compartiments

- Continuous vs. discrete models

  Modèles en temps continue vs. modèles en temps discrète

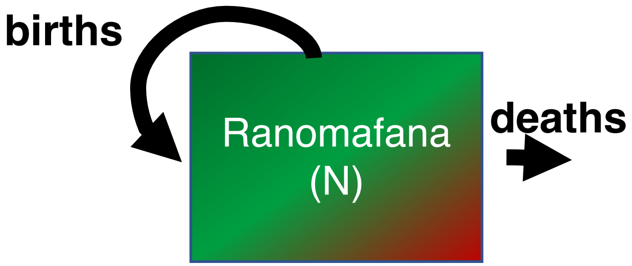
- Deterministic vs. stochastic models

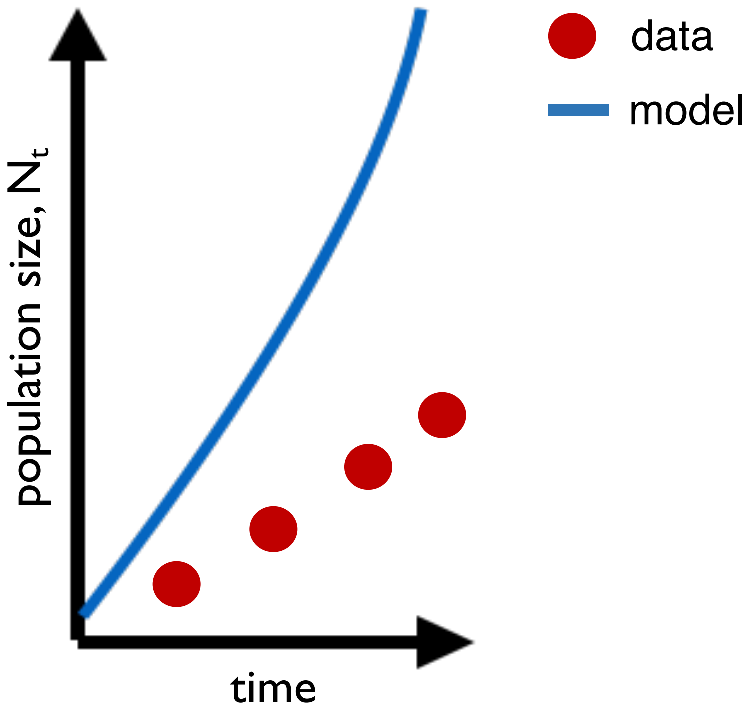
  Modèles déterministique vs. stochastique

2. Structured Population Models
2. Modèles de la population structurée

The structured population model

Why does the model perform poorly?





The basic population model

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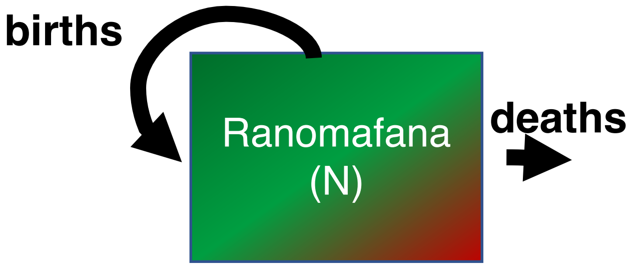
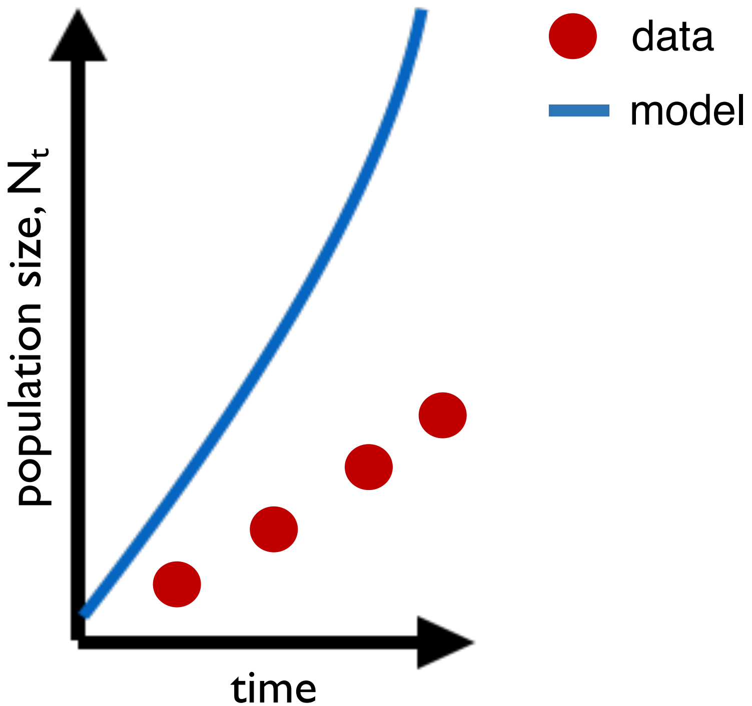




We need population structure!

The structured population model

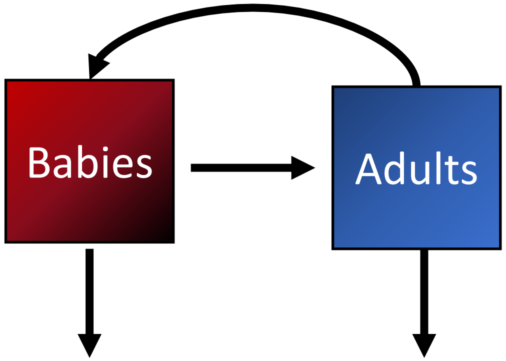
Compartmental models (Mechanistic Models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

How does the population of Ranomafana grow over time?

Comment est-ce que la population de Ranomafana s'augmente avec le passage du temps?

The structured population model



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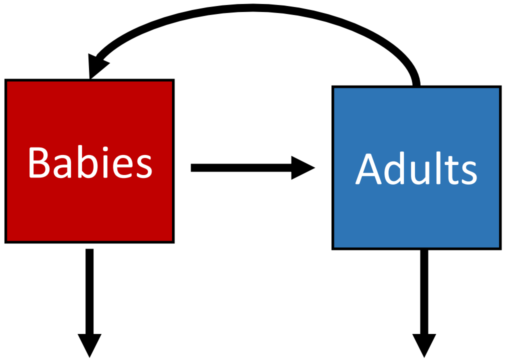

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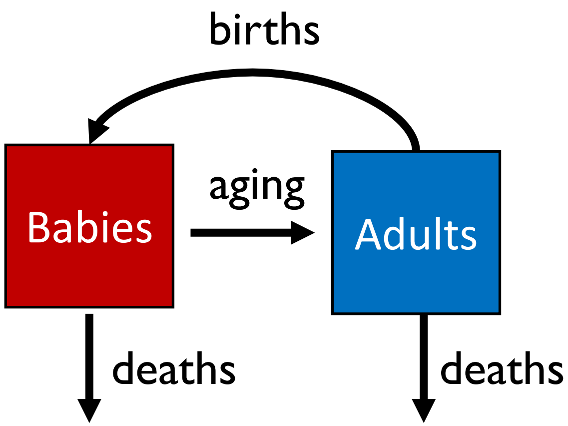

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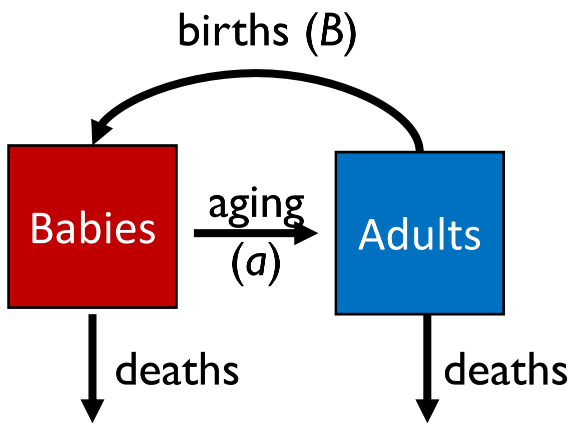

$$\lambda = N_{t+1}/N_t$$

pop size at t + 1

pop size at t

Population rate of increase

Taux d'accroissement de la population

Compartmental models (Mechanistic Models)

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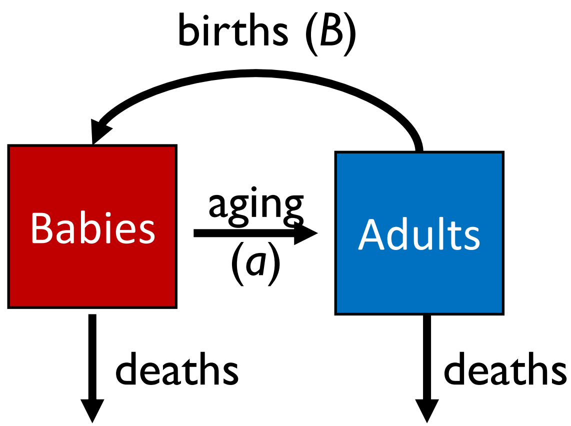
$$\mathbf{n_{t+1}} = \mathbf{A}\,\mathbf{n_t}$$

vector of population sizes

$$\begin{bmatrix} s_b(1-a) & B \\ s_b a & s_a \end{bmatrix}$$

The structured population model



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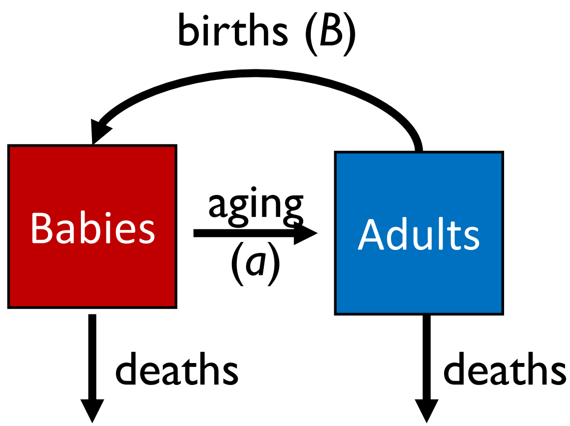
***Discrete time**

The structured population model



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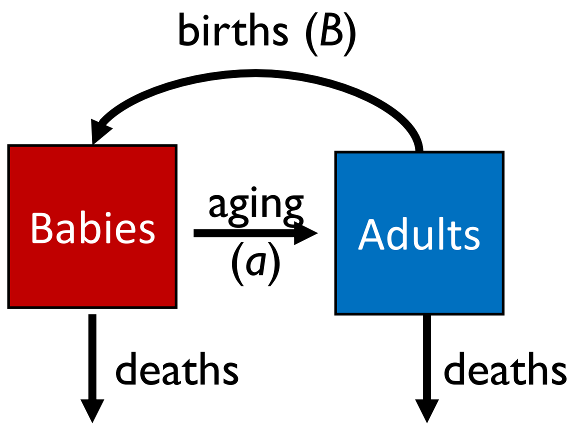
$$\overset{\mathbf{A}}{\begin{bmatrix} s_b(1-a) & B \\ s_b a & s_a \end{bmatrix}} \times \overset{\mathbf{n_t}}{\begin{bmatrix} n_b \\ n_a \end{bmatrix}} = \overset{\mathbf{n_{t+1}}}{\begin{bmatrix} s_b(1-a)\, n_b + b n_a \\ s_b\, a\, n_b + s_a n_a \end{bmatrix}}$$

The structured population model



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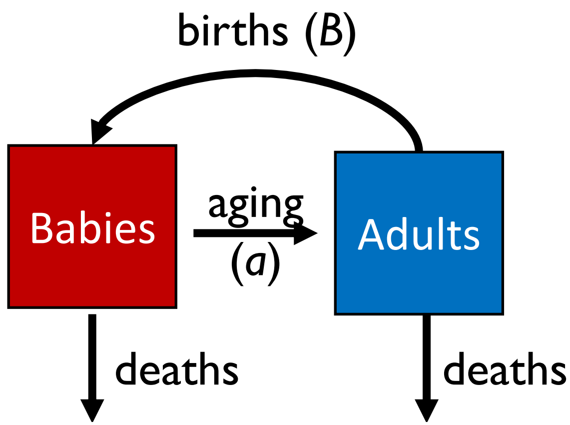
Population growth will depend on population structure!

The structured population model



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Conservation and Management of a Threatened Madagascar Palm Species, *Neodypsis decaryi*, Jumelle

JOELISOA RATSIRARSON,*‡ JOHN A. SILANDER, JR.,* AND ALISON F. RICHARD†

*Department of Ecology and Evolutionary Biology, 75 N. Eagleville Road, The University of Connecticut, Storrs, CT 06269, U.S.A.

†Yale School of Forestry and Environmental Studies, 205 Prospect Street, New Haven, CT 06520, U.S.A.

‡Current Address: Yale School of Forestry and Environmental Studies, 205 Prospect Street, New Haven, CT 06520, U.S.A.

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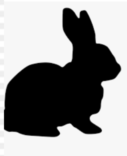

3. Two-population model
3. modèles de deux populations

The predator-prey model

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How does the population of fossa regulate the population of lemurs in Ranomafana?

Comment la population de “fossa” régule la population de lemuriens à Ranomafana?

The predator-prey model

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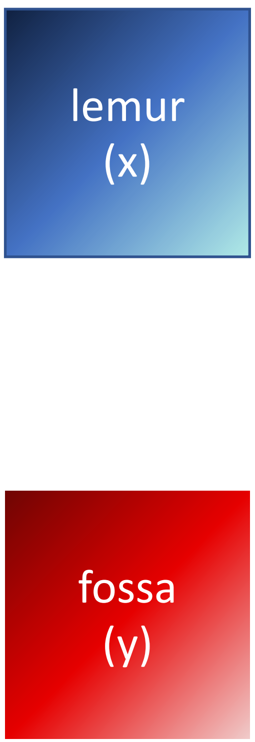

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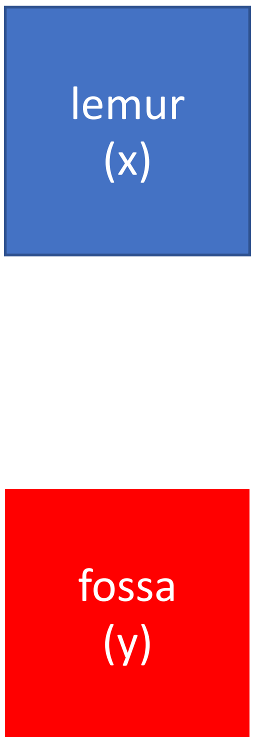

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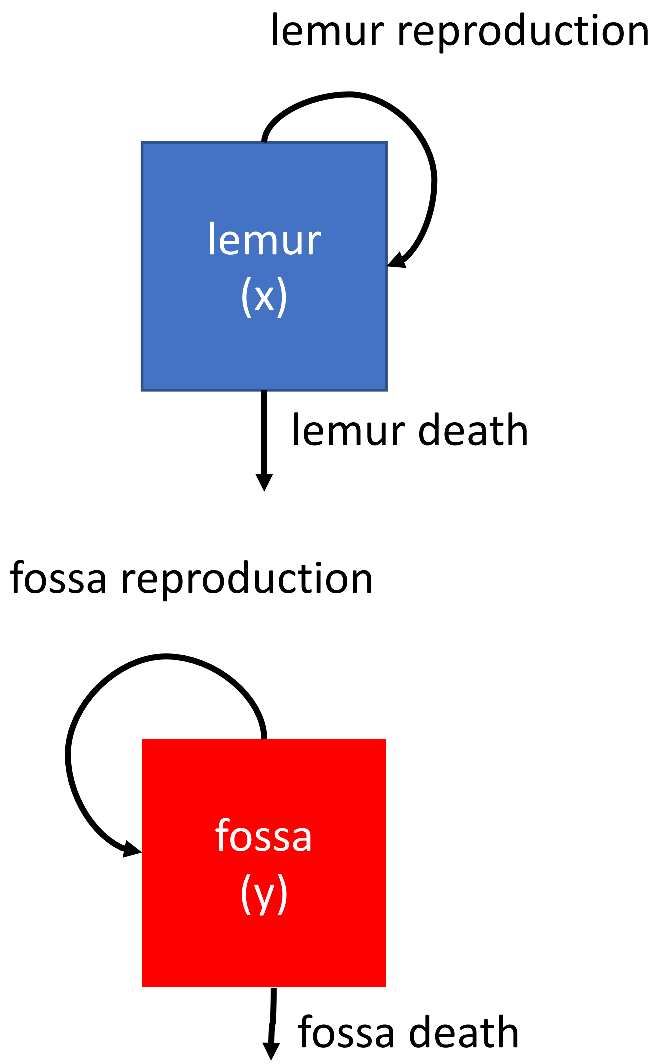

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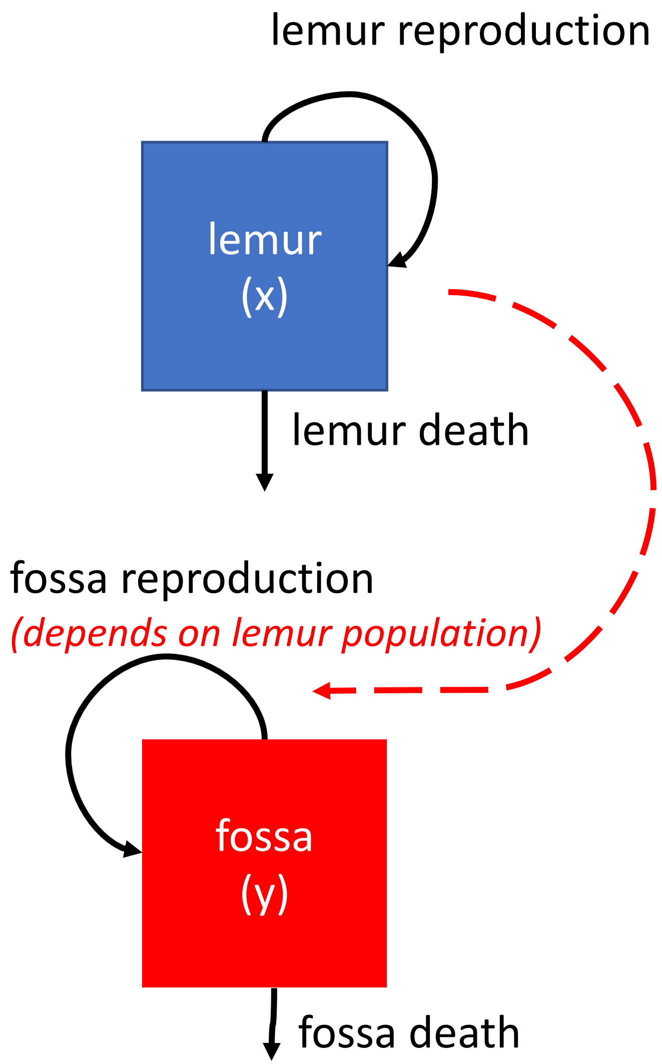

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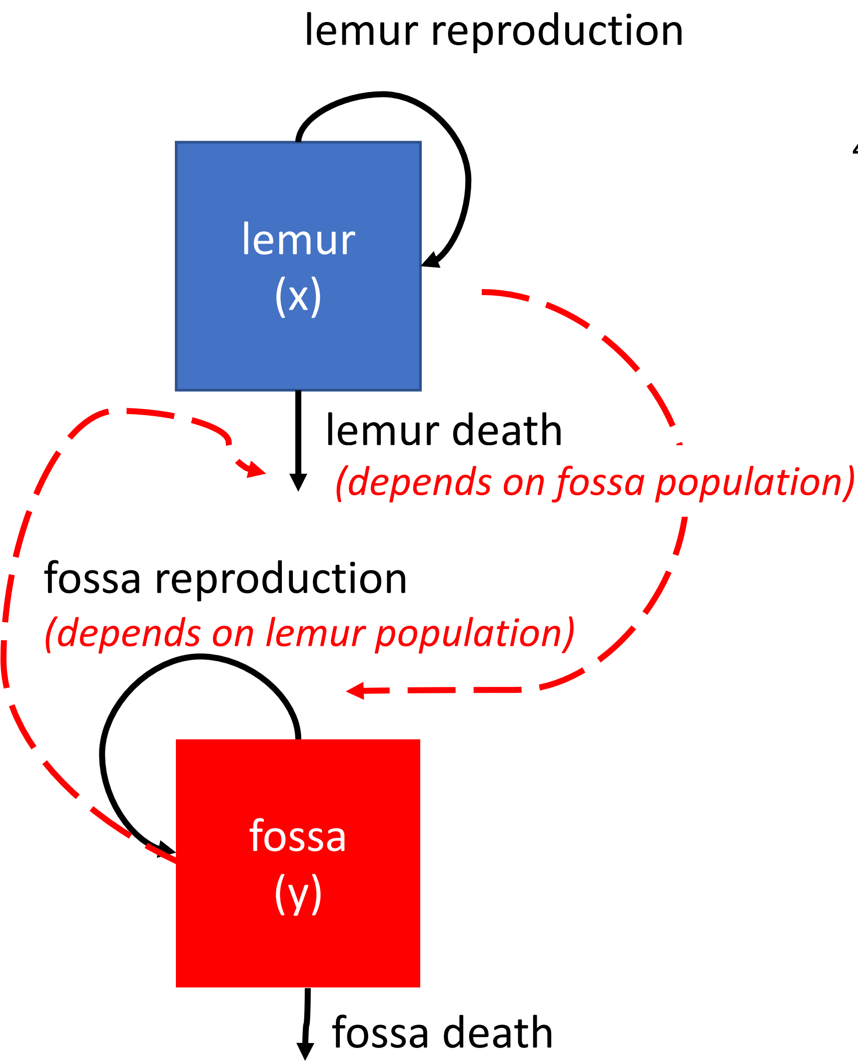

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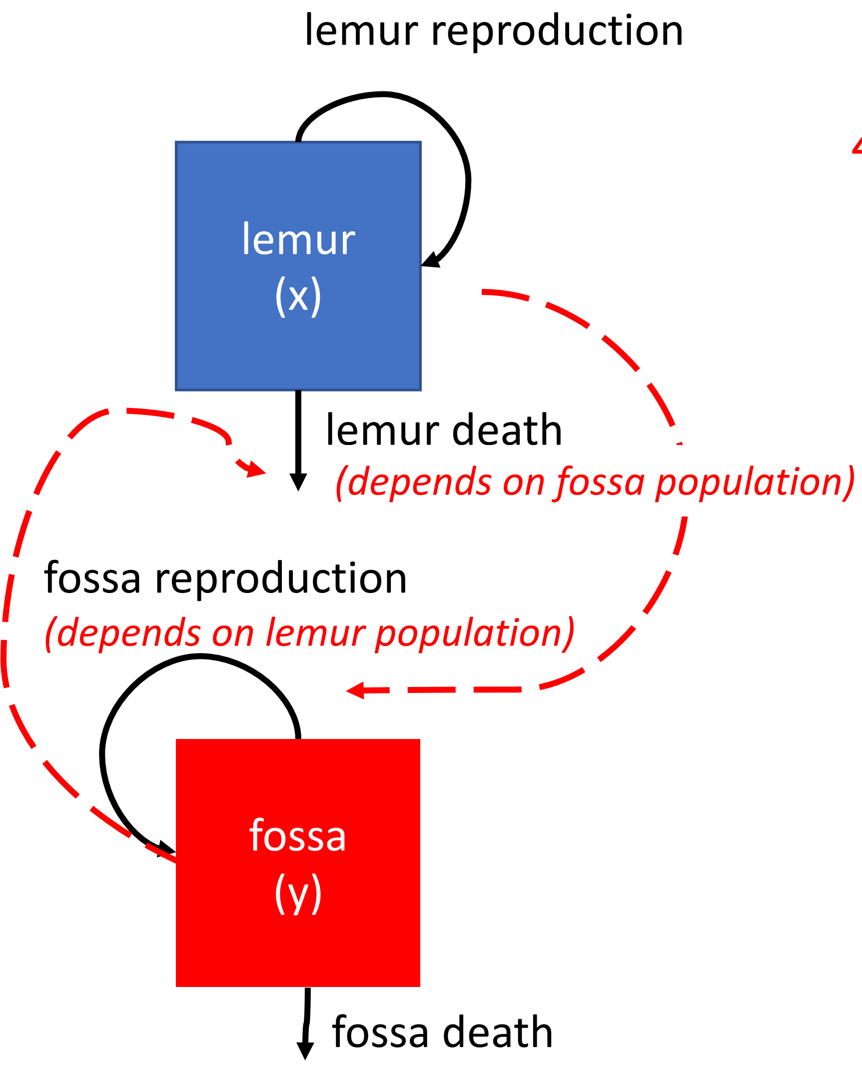

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Parameters

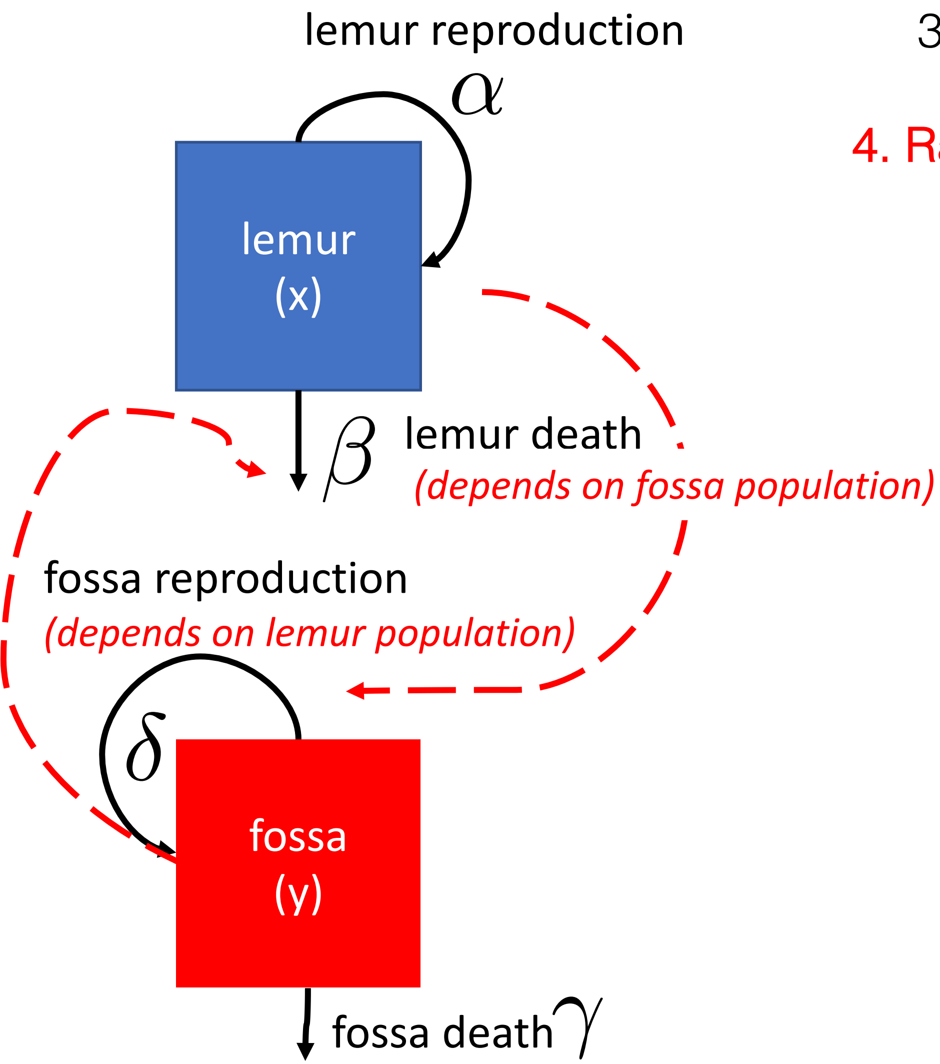
: lemur rep. rate

: lemur death rate

: fossa rep. rate

: fossa death rate

The predator-prey model



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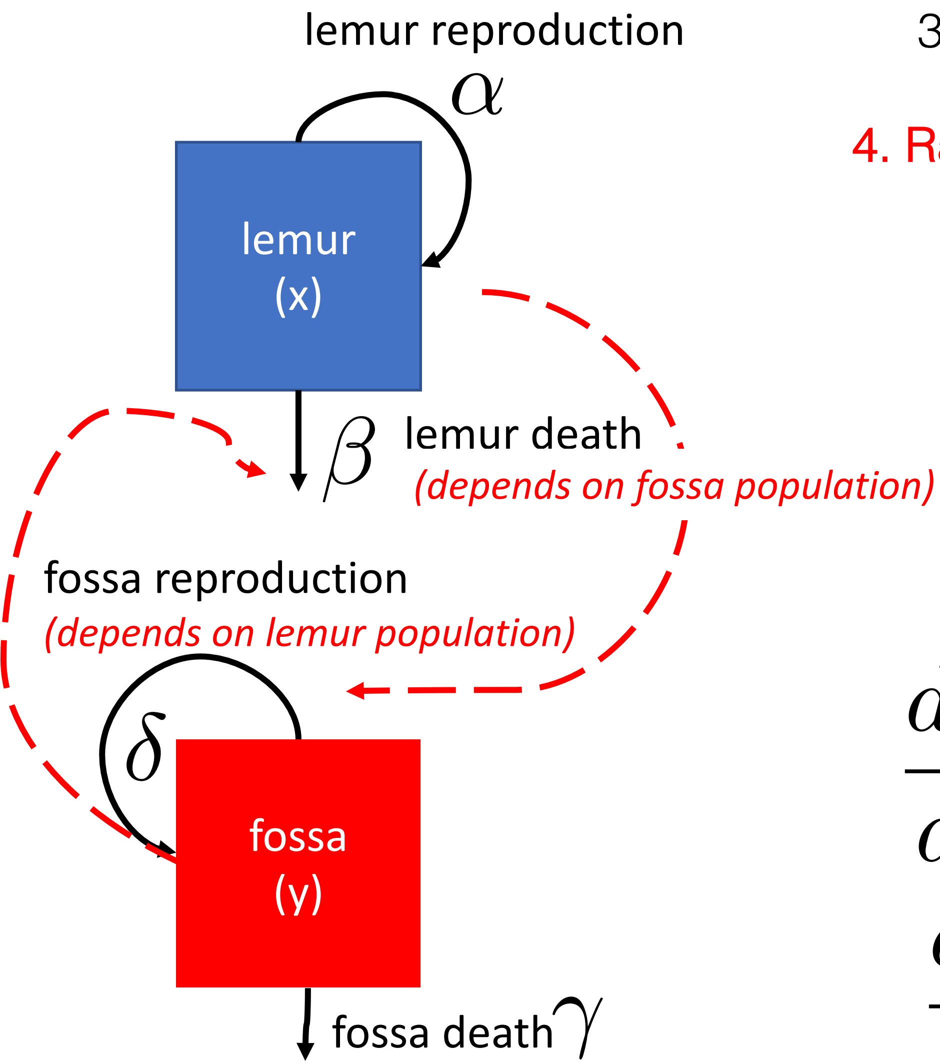
β : lemur death rate

δ : fossa rep. rate

γ : fossa death rate

The predator-prey model



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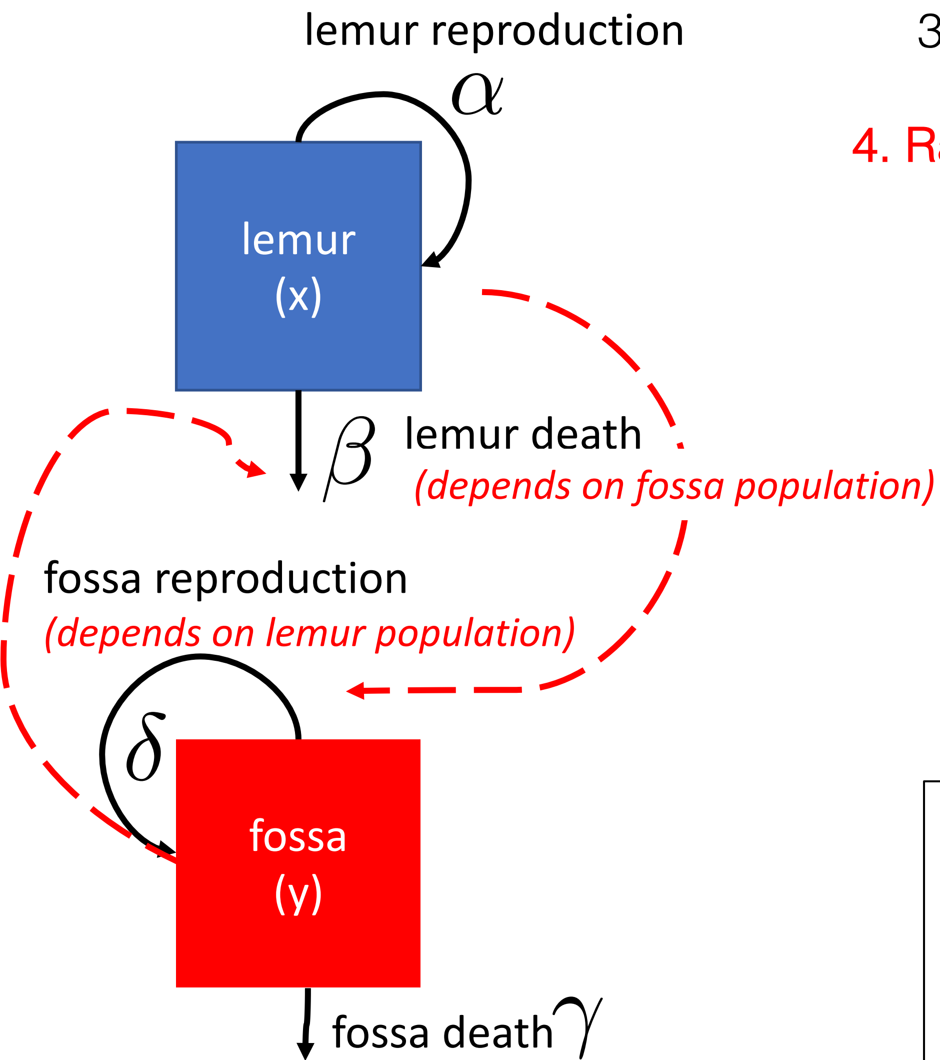
$$\frac{dx}{dt} = x(\alpha - \beta y)$$

$$\frac{dy}{dt} = y(\delta x - \gamma)$$

The predator-prey model



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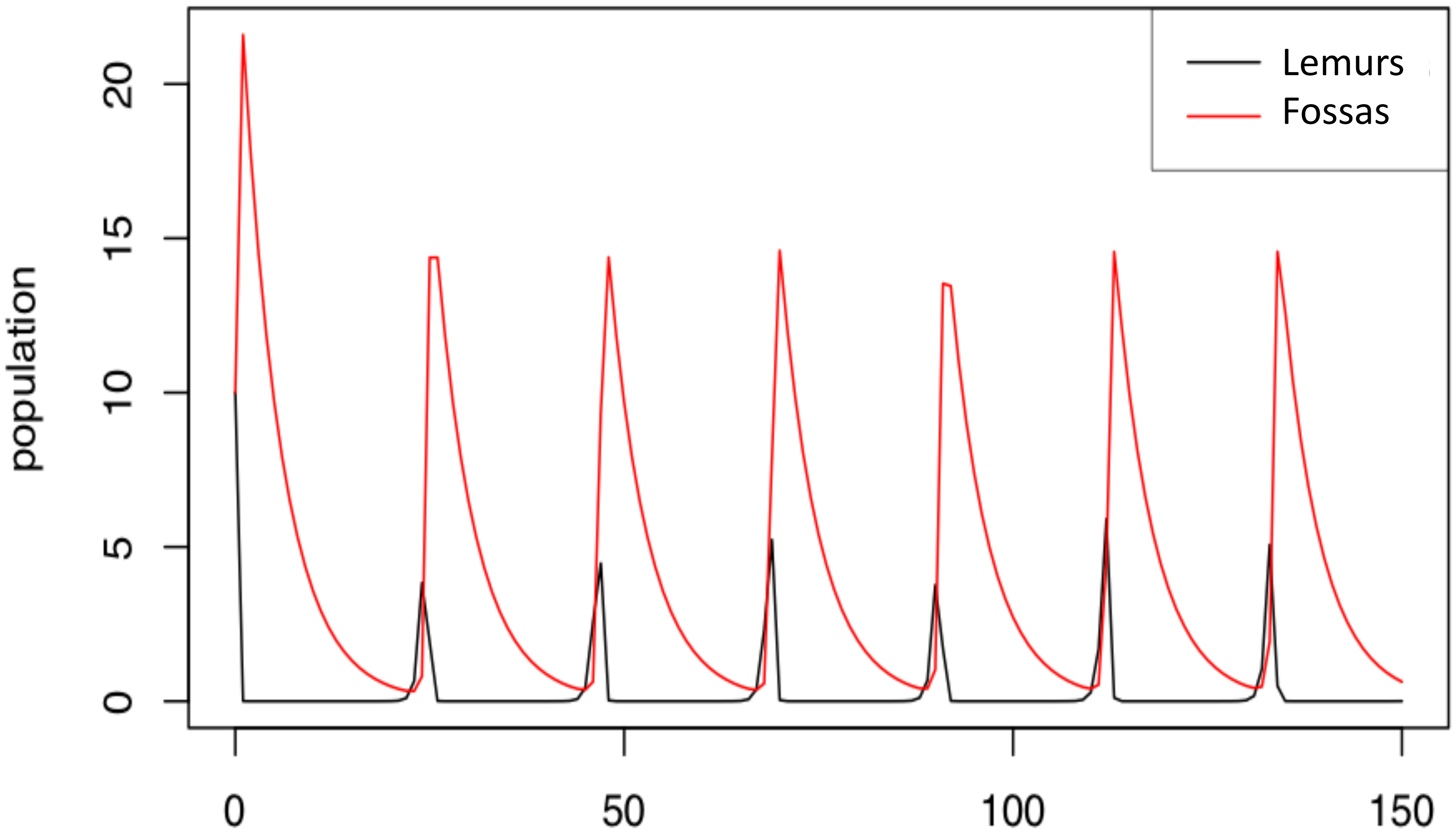
SOME ASSUMPTIONS

- the **lemur** has an unlimited food supply
- the **lemur** only dies from being eaten by fossa
- the **fossa** is totally dependent on a single prey species (the lemur) as its only food supply

The predator-prey model



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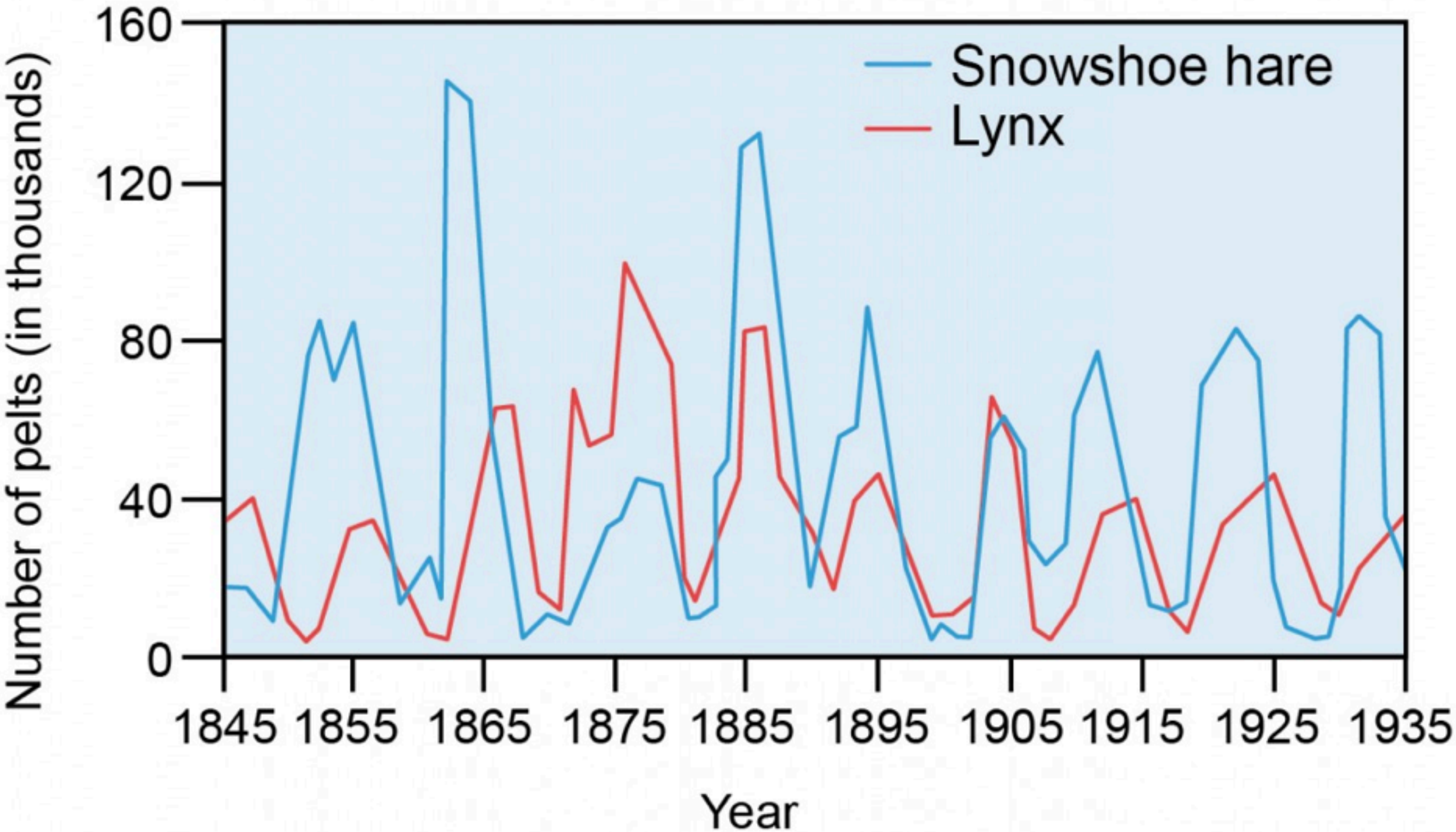

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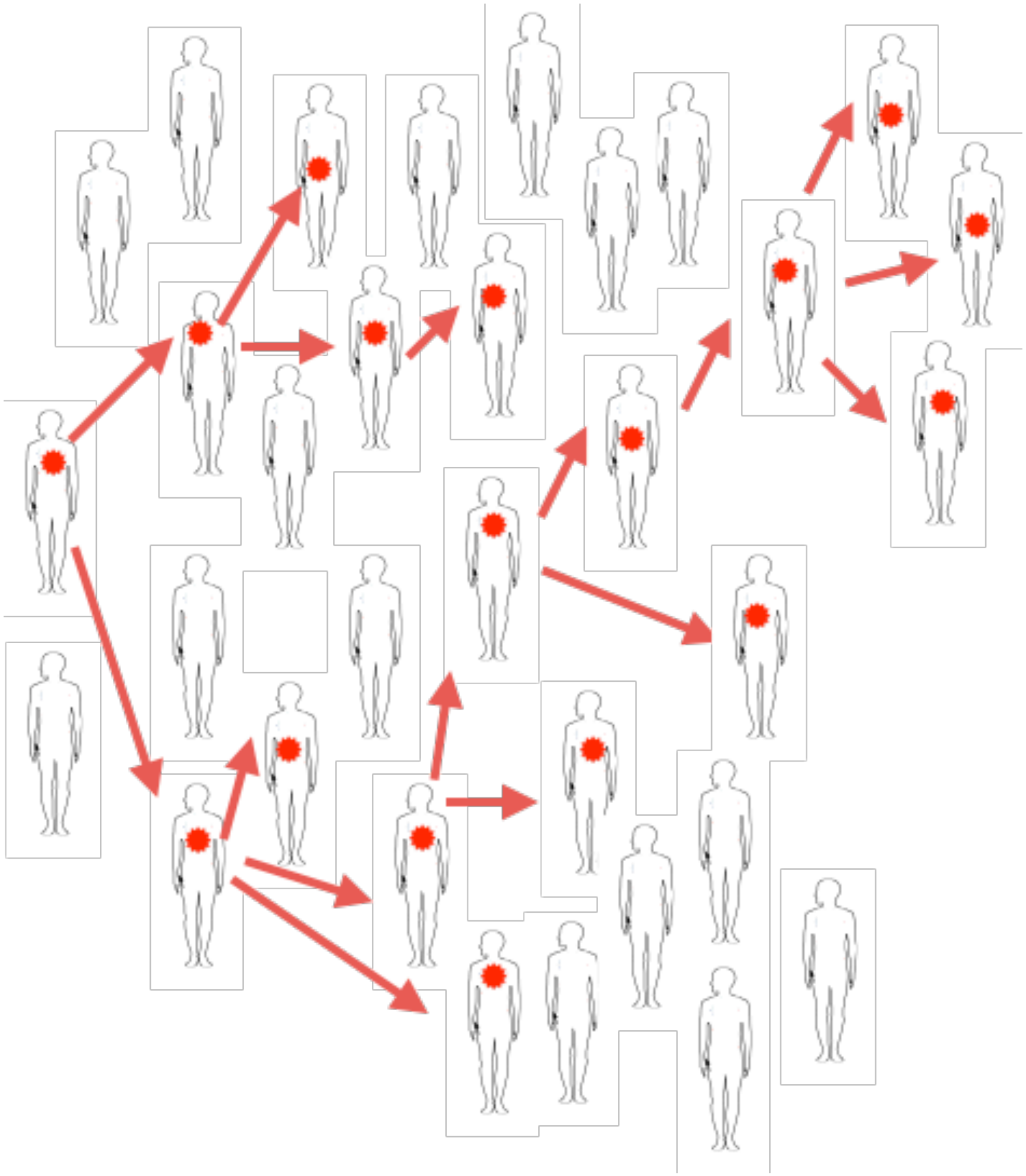
  Modèles des deux populations

4. SIR Models
4. Les modèles SIR



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How does measles transmit through Antananarivo?

Comment la rougéole se transmet-elle à Antananarivo?

The SIR model

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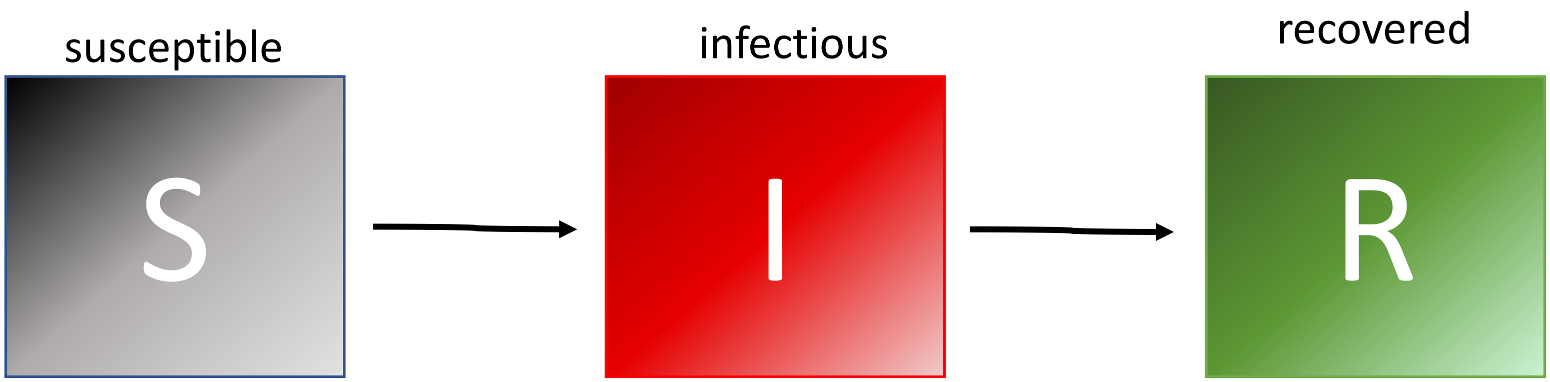

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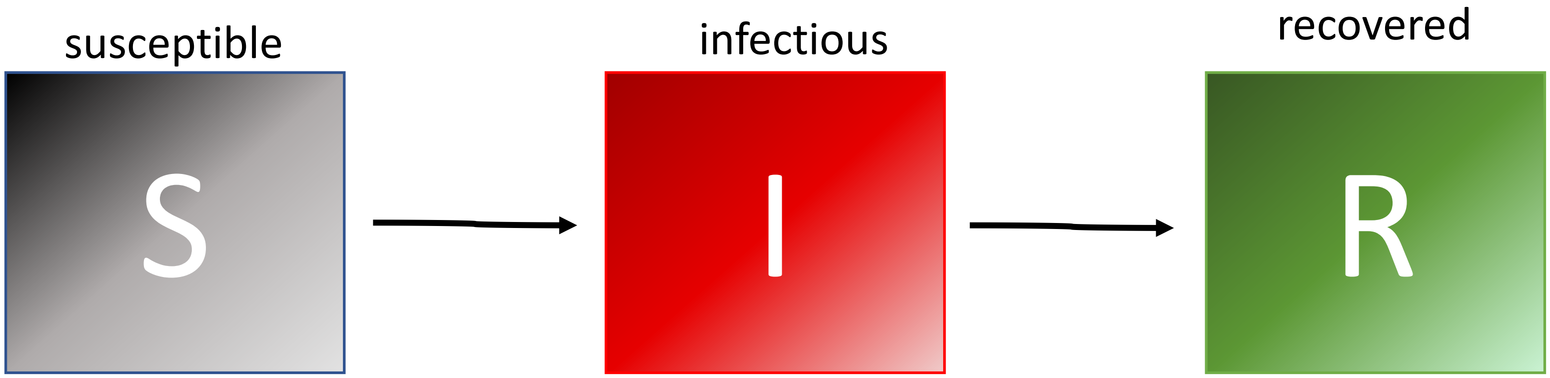

What are the big assumptions here?

The SIR model

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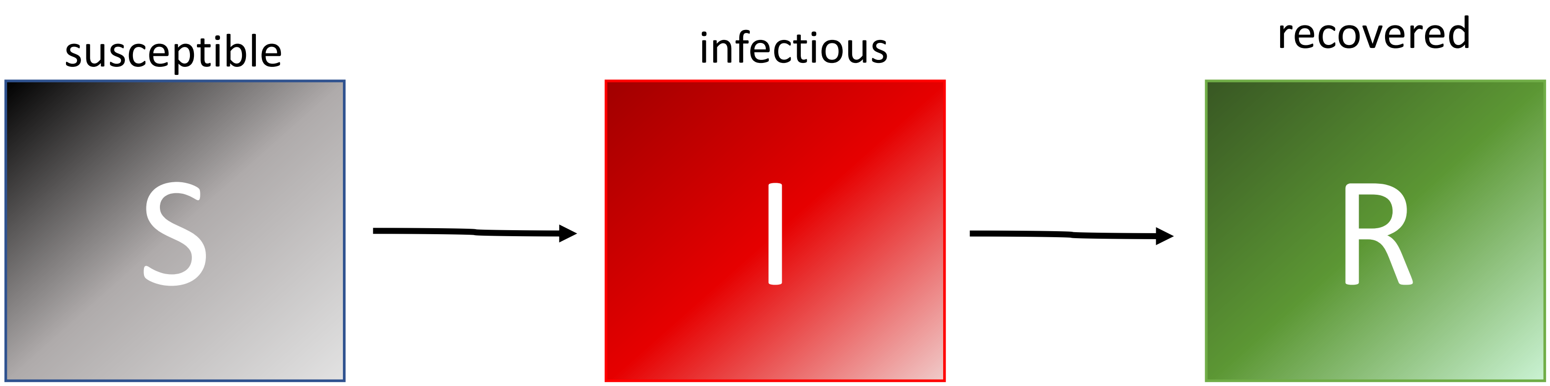
everyone is either:



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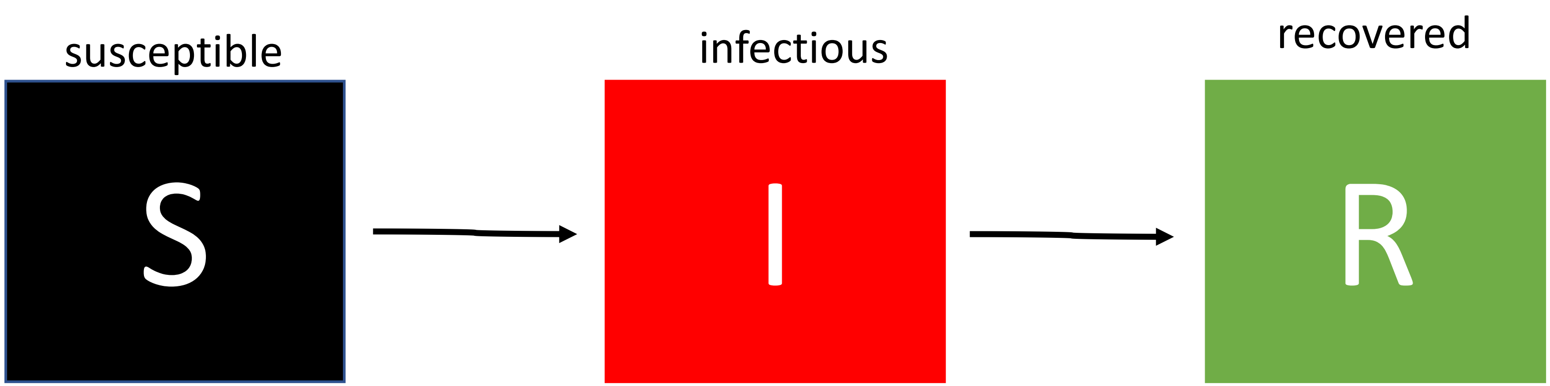

people mix uniformly (mass action)

les gens se mélangent uniformément

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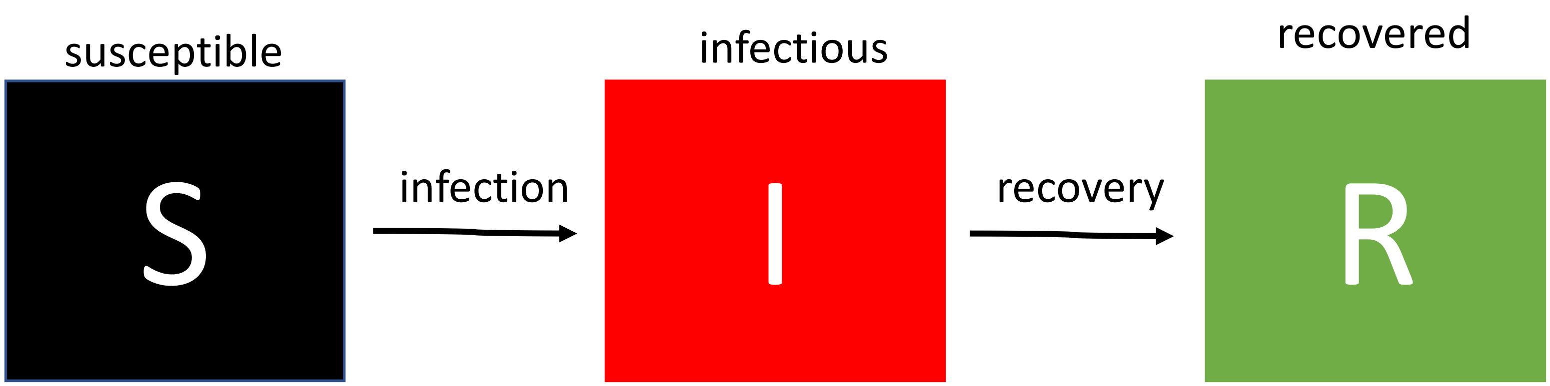

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les gens se mélangent uniformément

no latent period
(infectious when infected)
pas de période de latence

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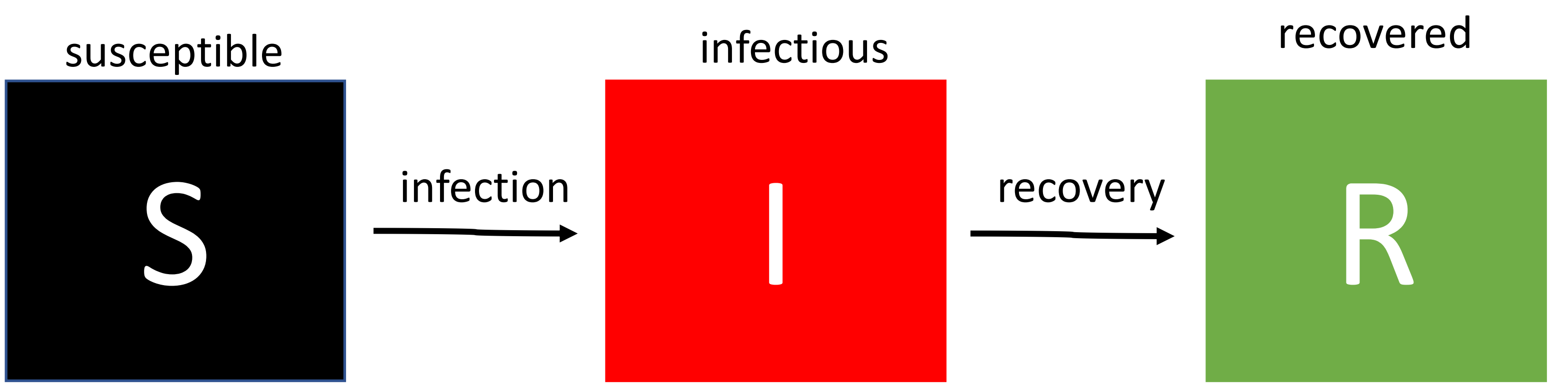

recovery is permanent

la récupération est permanente

people mix uniformly (mass action)

les gens se mélangent uniformément

no latent period (infectious when infected)

pas de période de latence

The SIR model

population size constant - no births or deaths, migration

taille de population constante

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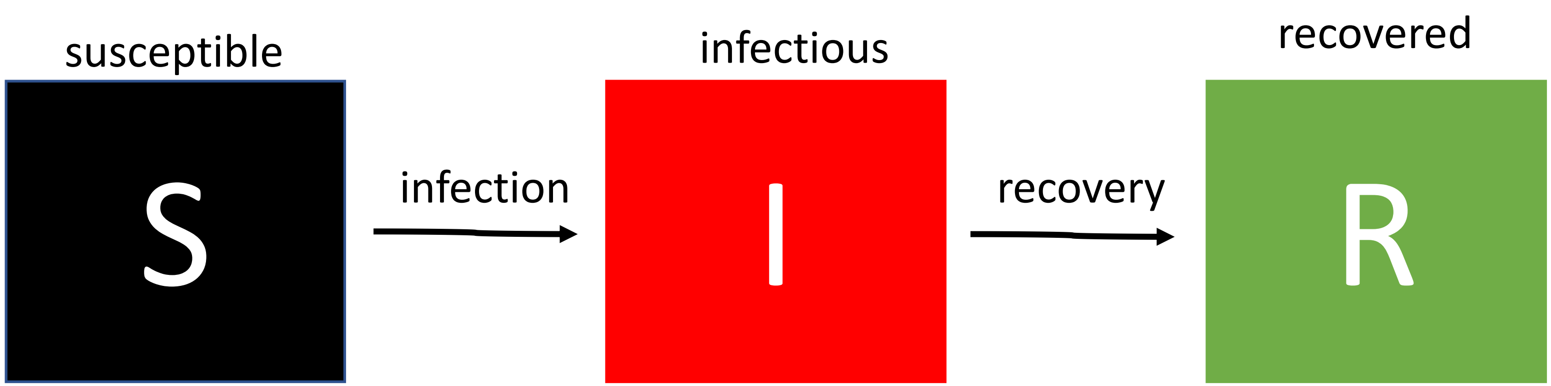

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Parameters

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γ: rate of recovery

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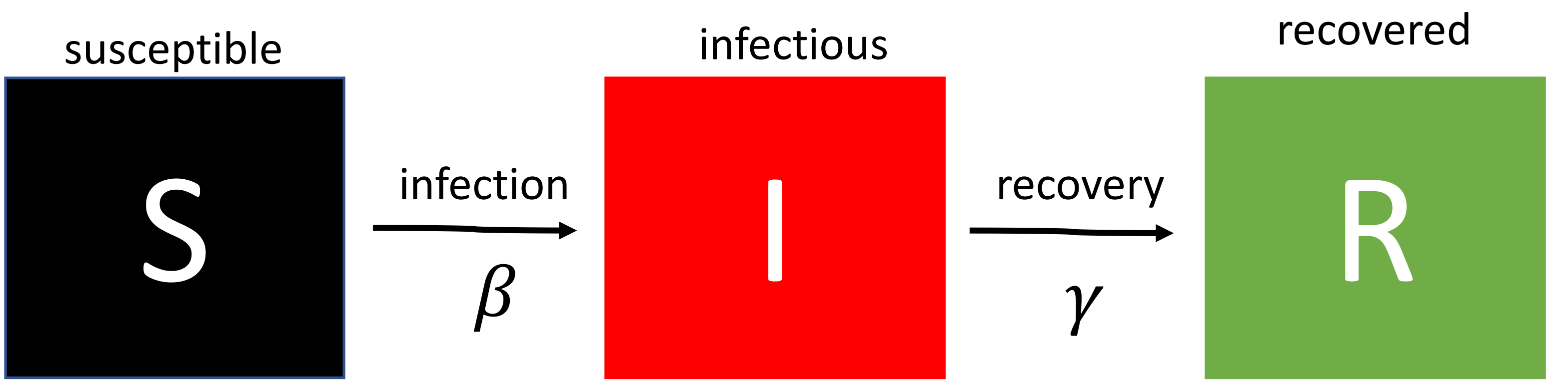

$$\frac{dS(t)}{dt} = -\beta S(t)I(t)$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t)$$

The SIR model

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Parameters
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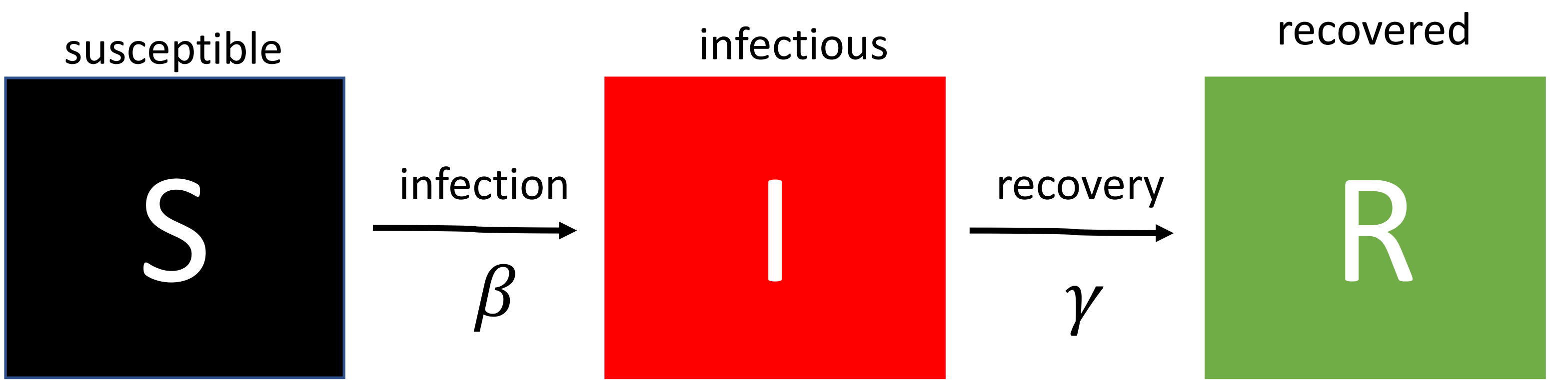


$$\frac{dS(t)}{dt} = -\beta \boxed{S(t)} I(t)$$

$$\frac{dI(t)}{dt} = \beta S(t) I(t) - \gamma I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t)$$

…multiply rates by box you start in….

The SIR model

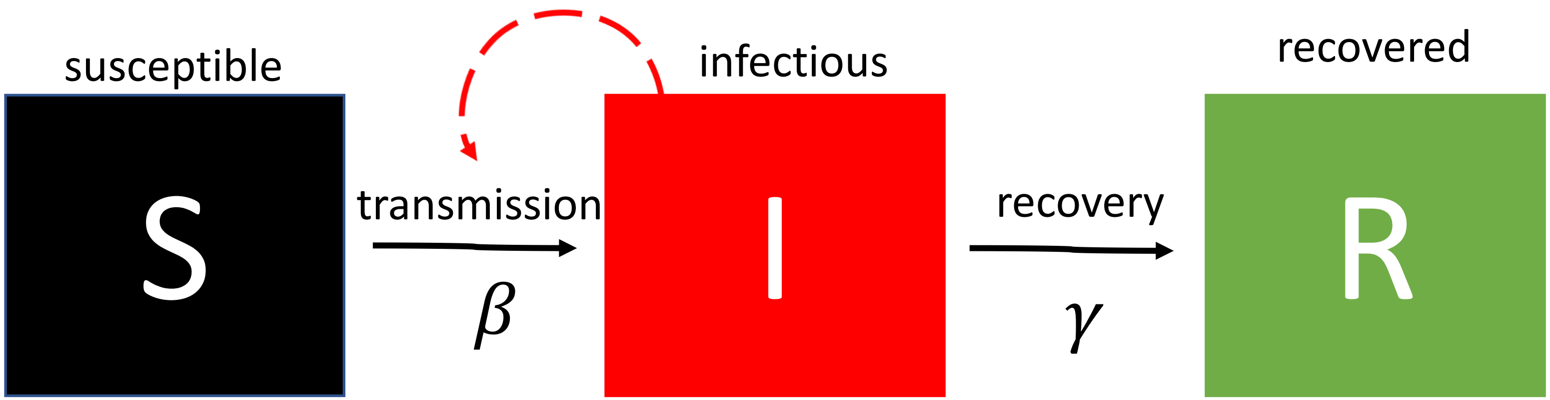
Compartmental models (Mechanistic Models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

Parameters

β: transmission rate

γ: rate of recovery



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$$\frac{dR(t)}{dt} = \gamma I(t)$$

…infected numbers influence the transmission rate….

The SIR model

Parameters

β: transmission rate

γ: rate of recovery

Compartmental models (Mechanistic Models)

1. Populations are divided into compartments
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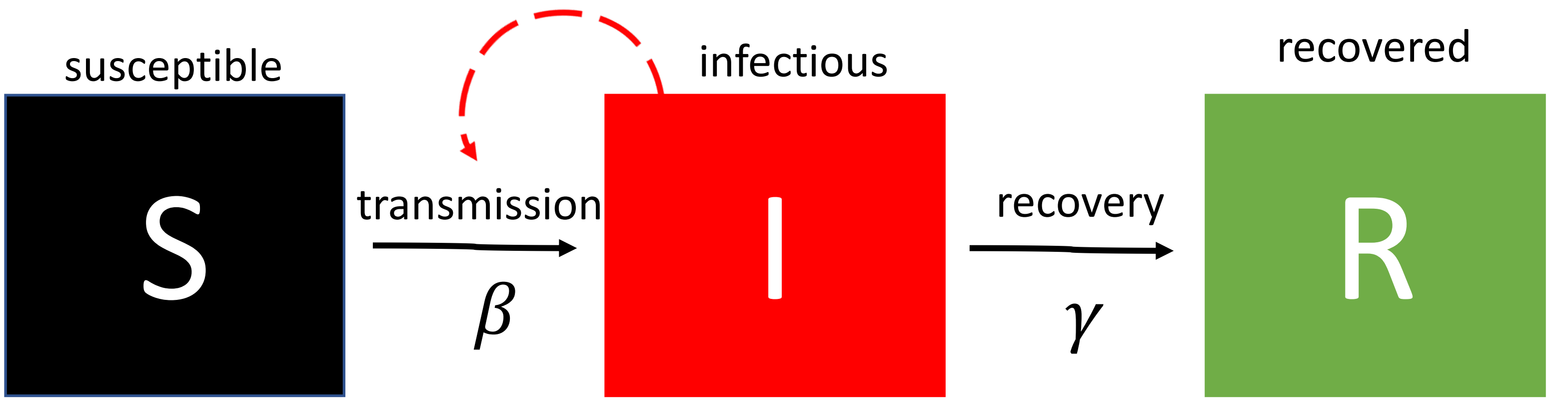
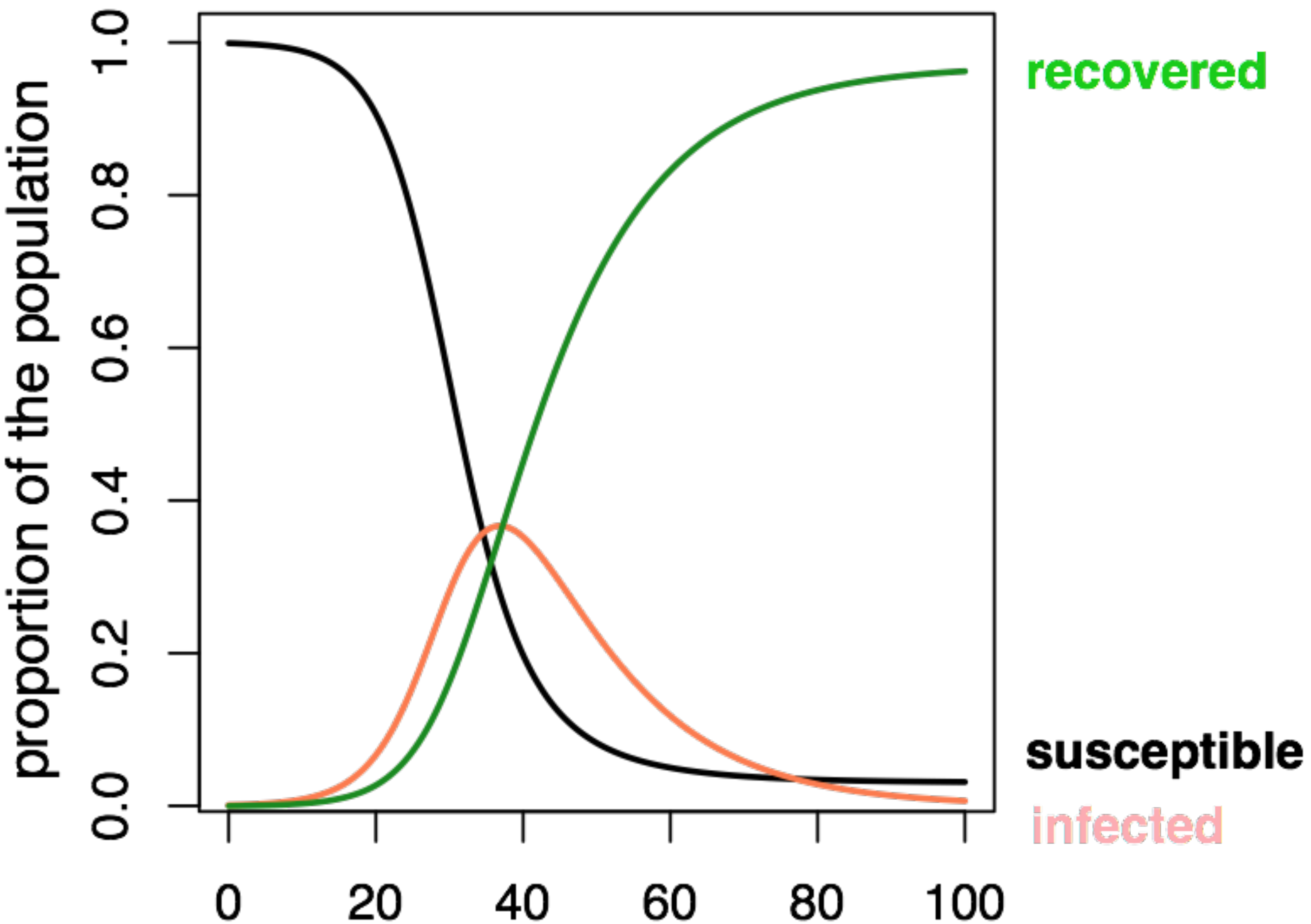
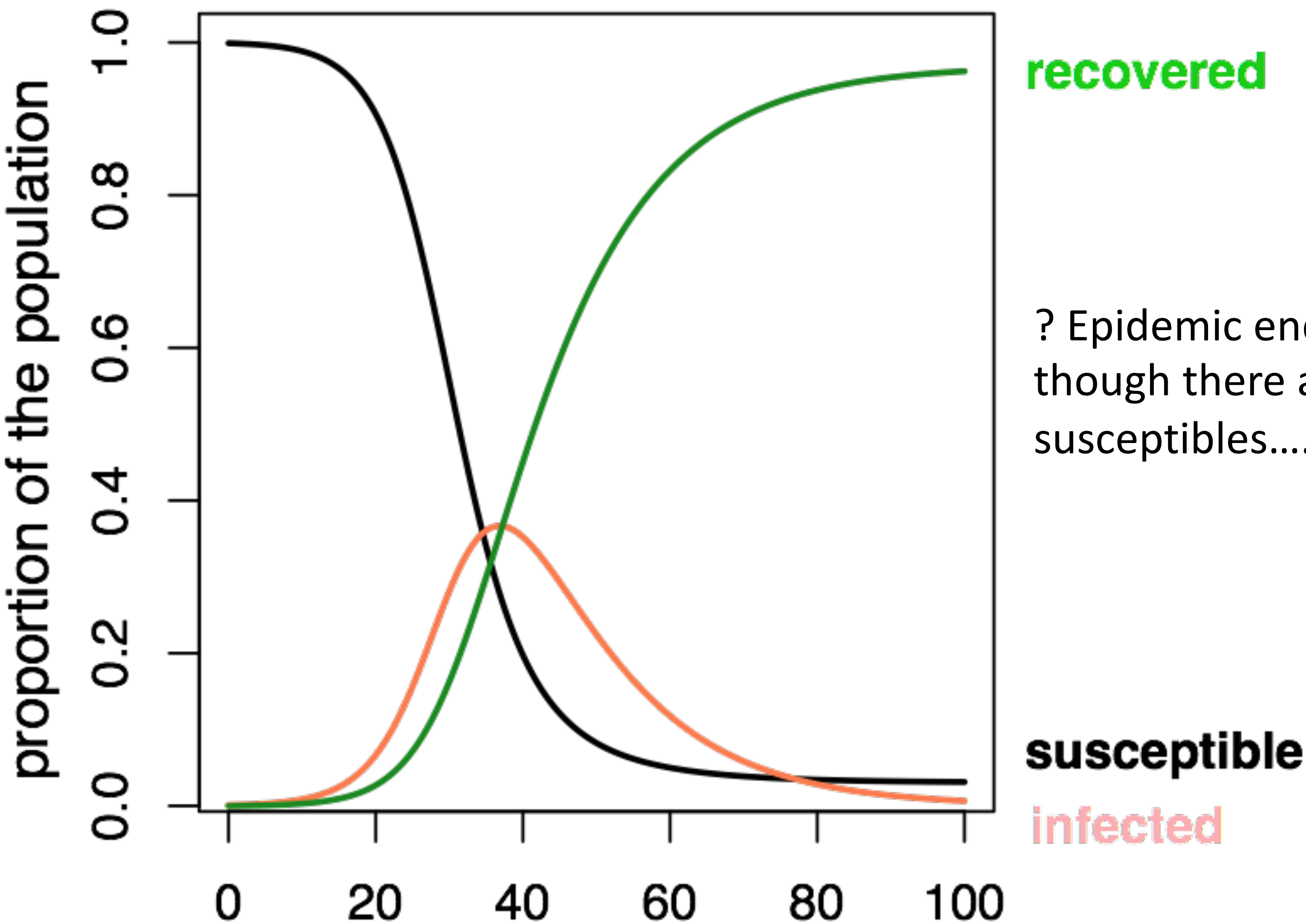


What will the dynamics look like?

The SIR model

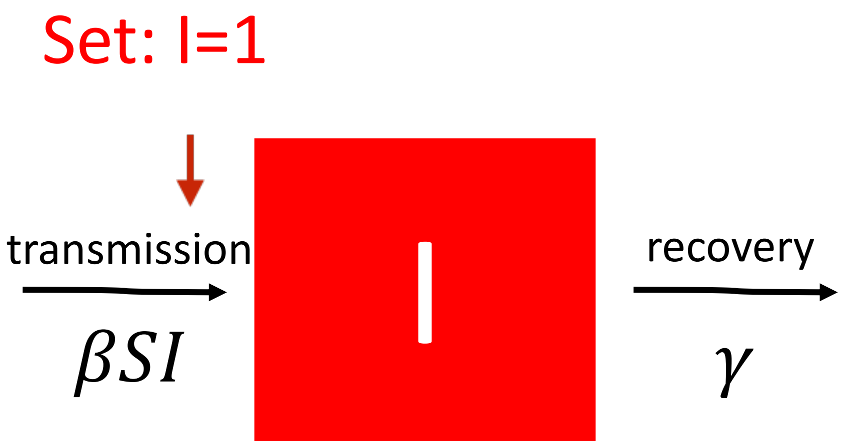


The SIR model



? Epidemic ends even though there are still some susceptibles….HOW?

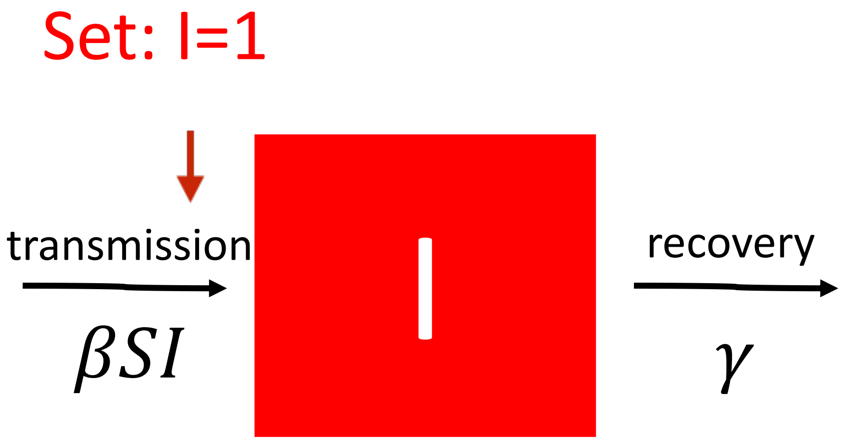
The SIR model

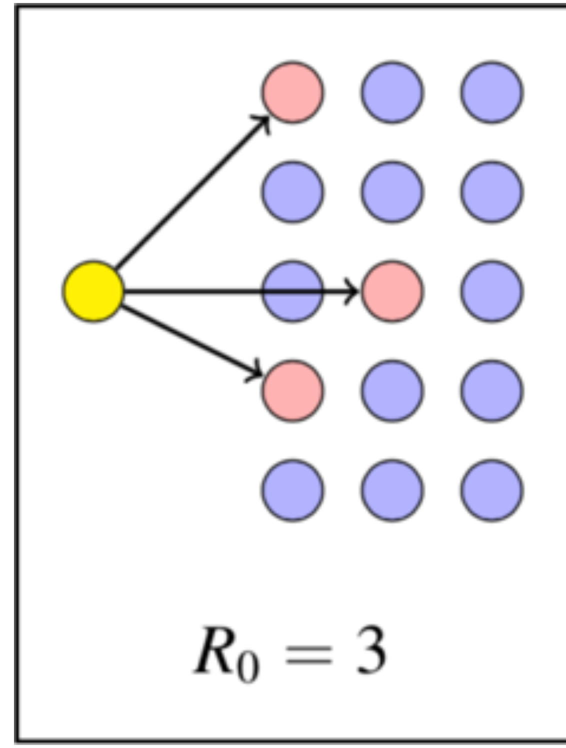


$$R_0 = \beta N / \gamma$$

The average number of persons infected by an infectious individual when everyone is susceptible (S=100%, or S=1, start of an epidemic)

The SIR model





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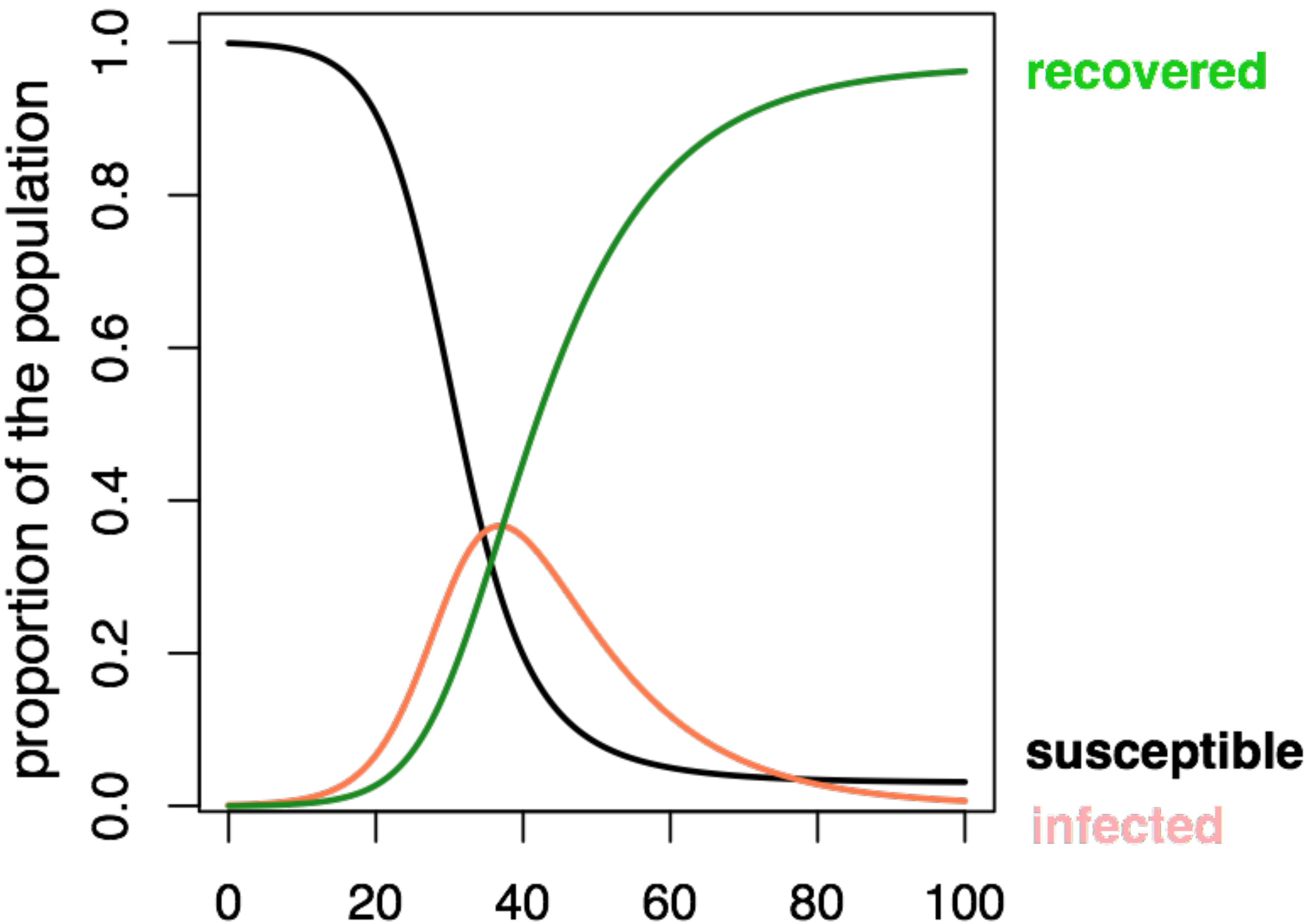

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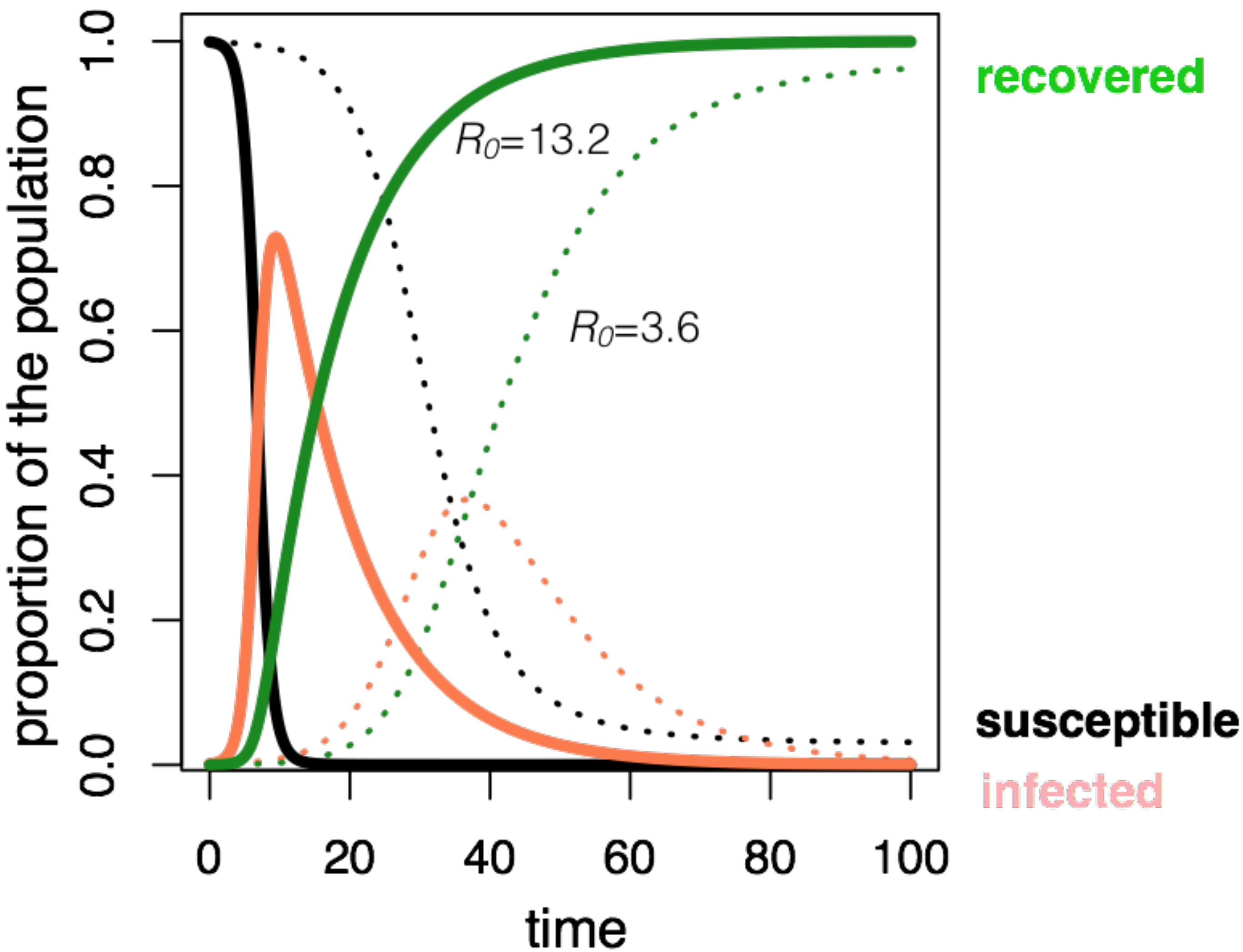

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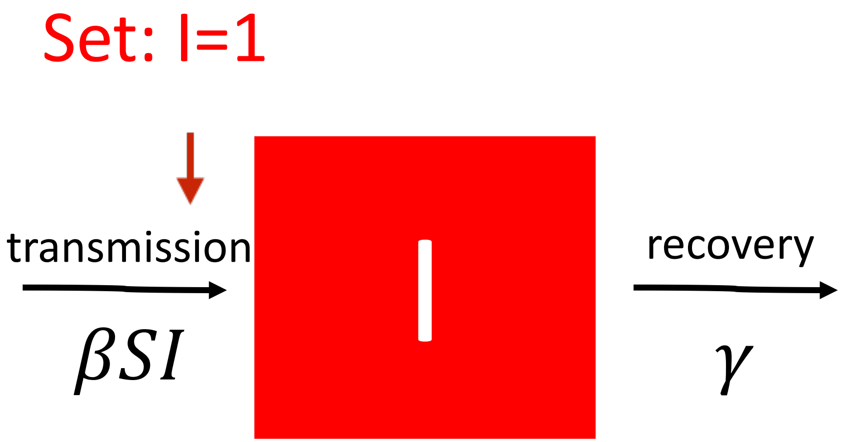


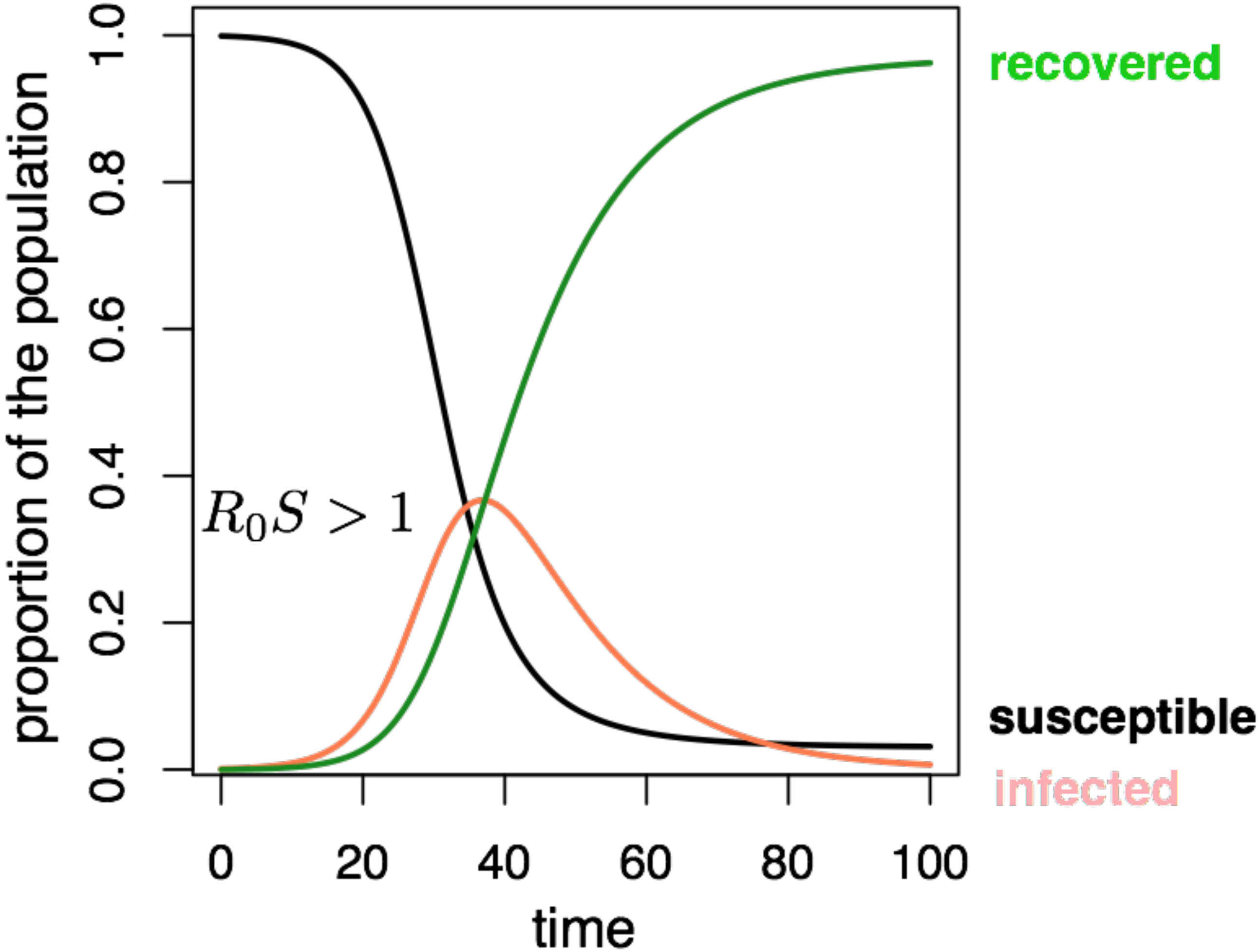
$$R_0 = \beta N / \gamma$$

The average number of persons infected by an infectious individual when everyone is susceptible (S=100%, or S=1, start of an epidemic)

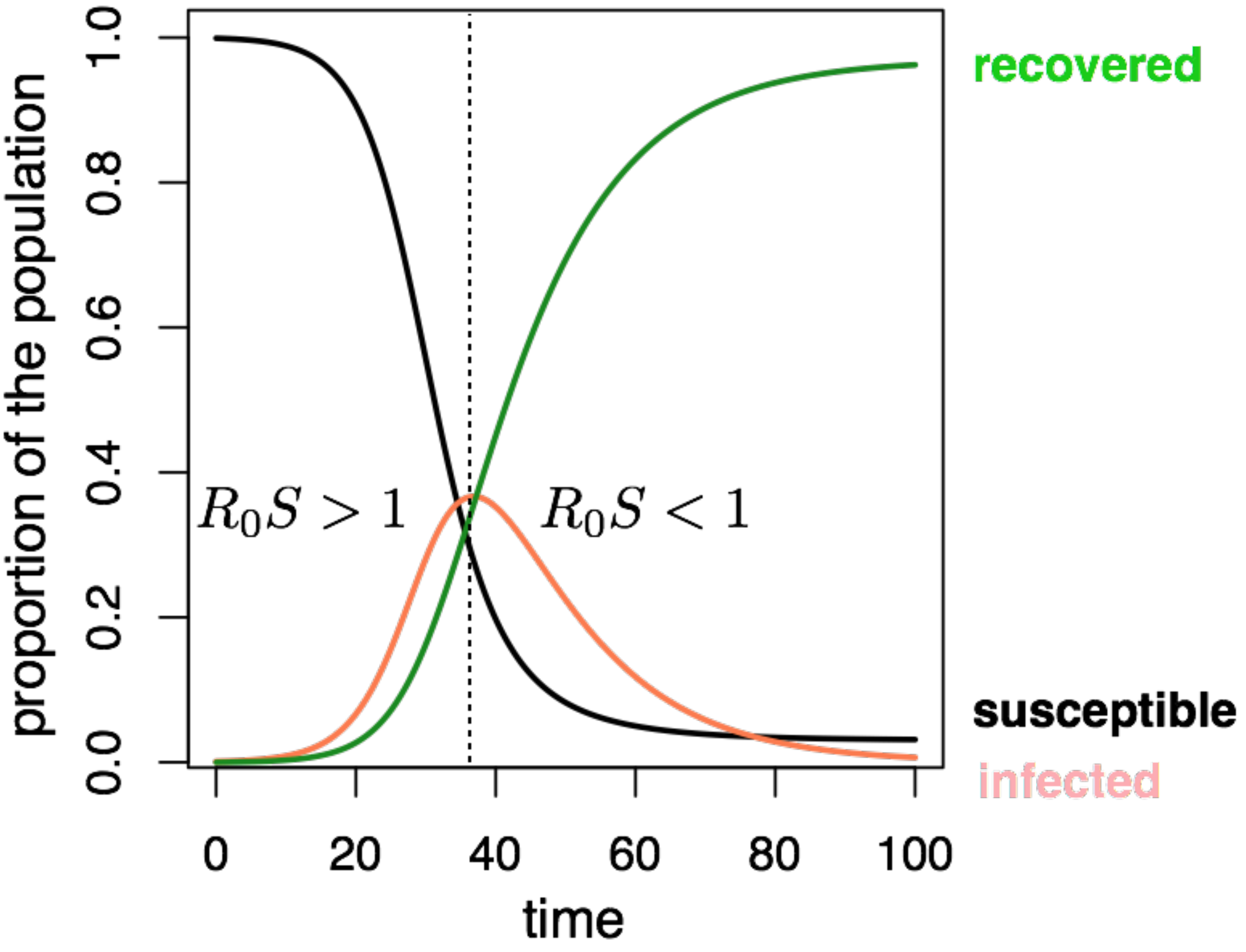
$$R_E = R_0 S$$ “R-effective"

…as the epidemic progresses and S falls

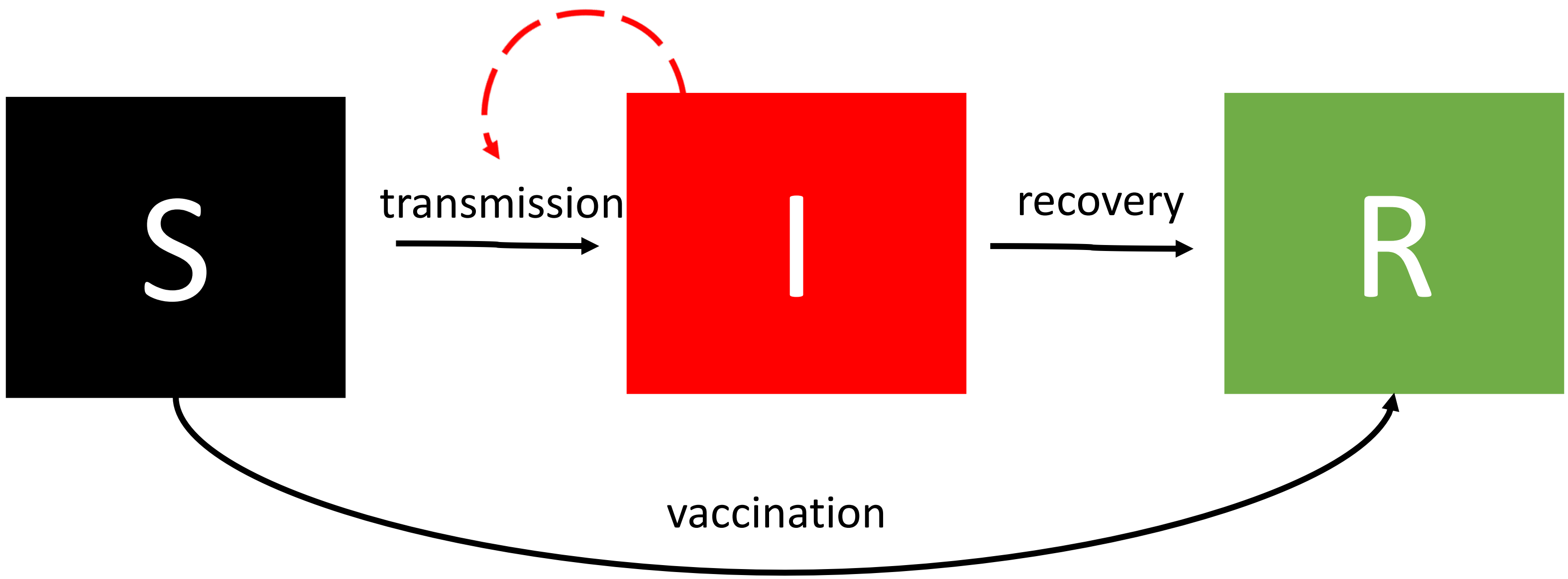
The SIR model



The SIR model



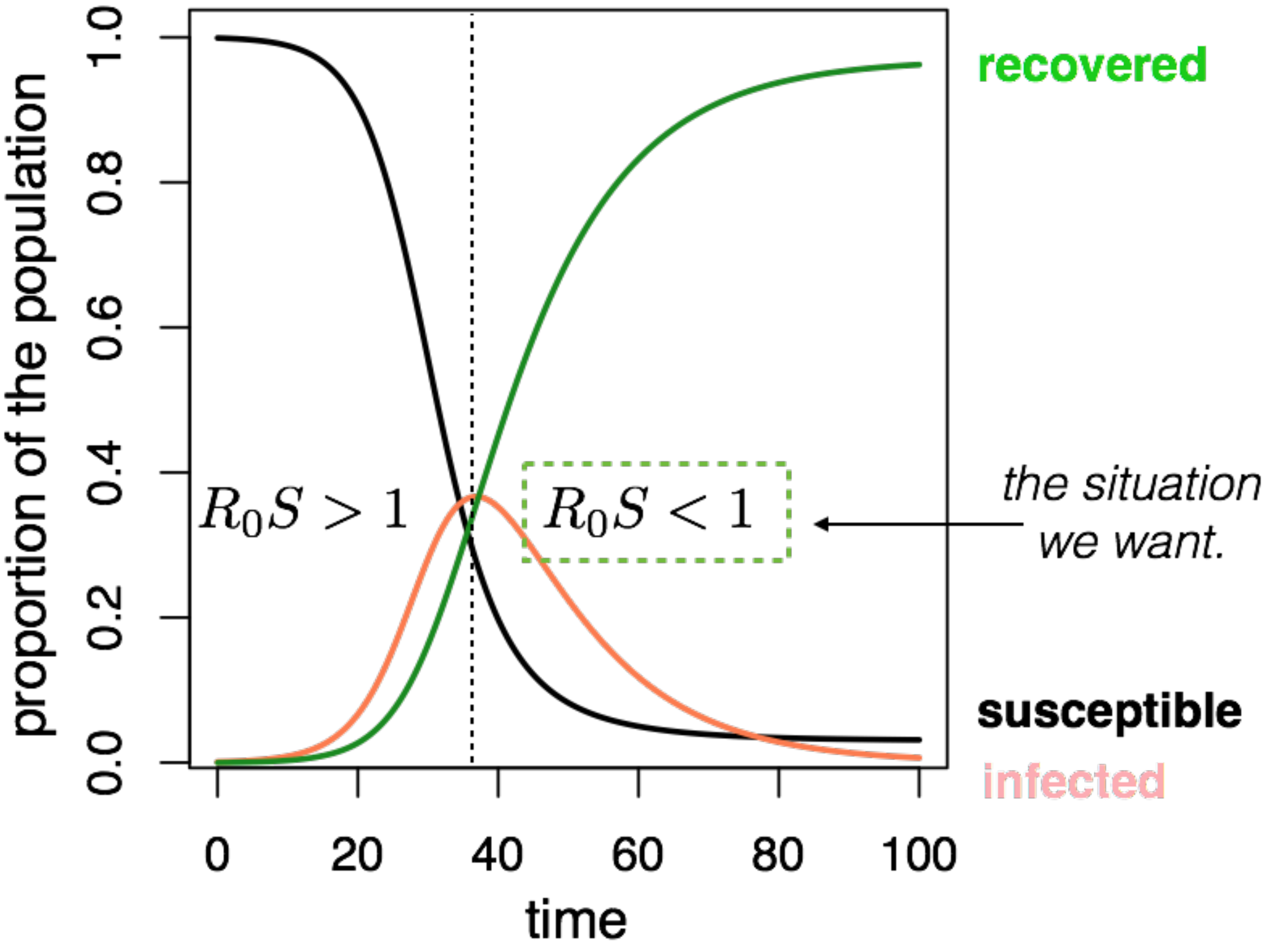
The SIR model : vaccination



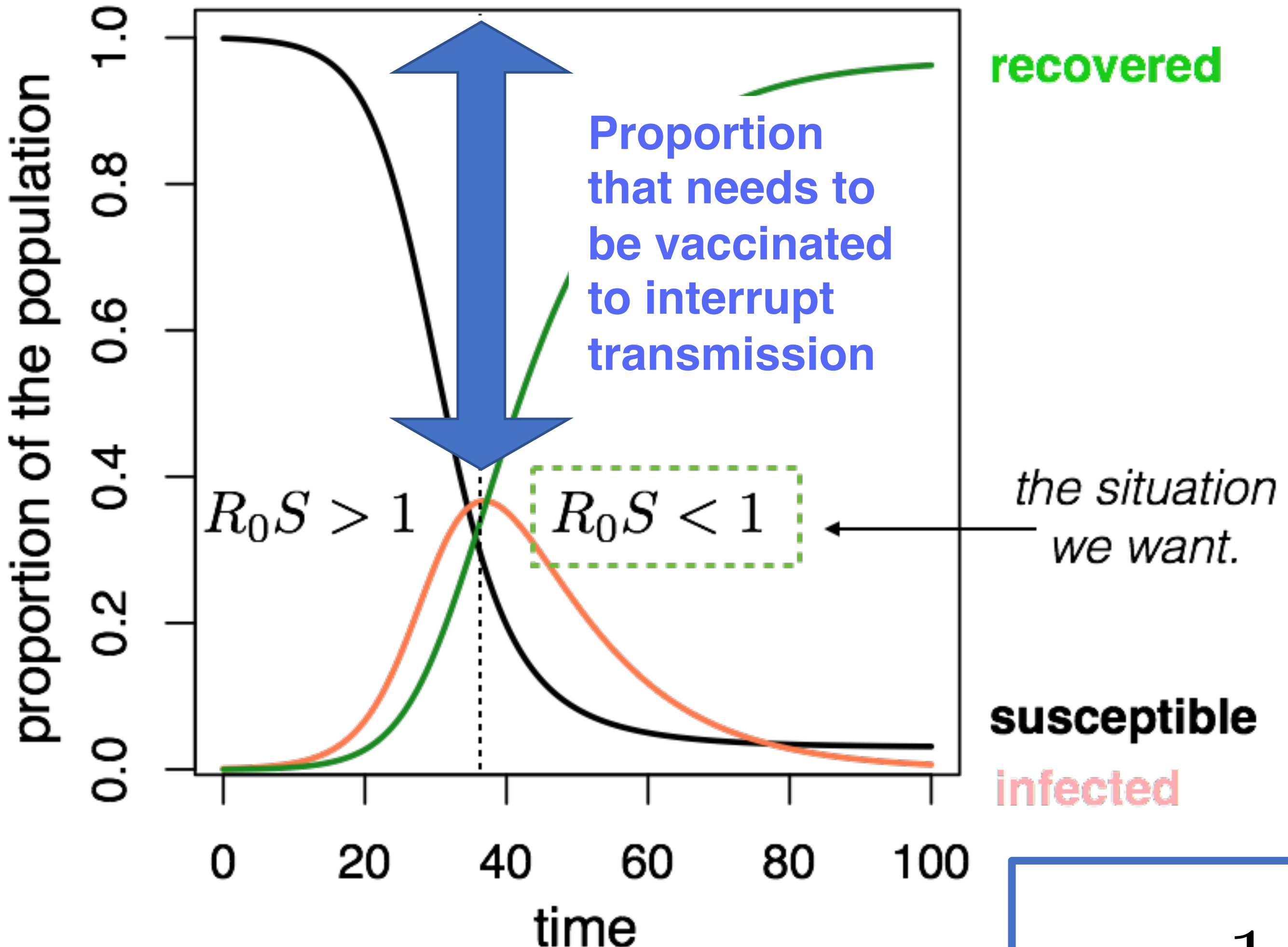
Vaccination moves people out of susceptibles into the immune (recovered) class.

La vaccination éloigne les personnes sensibles de la maladie dans la classe immunitaire (rétablie).

The SIR model : vaccination



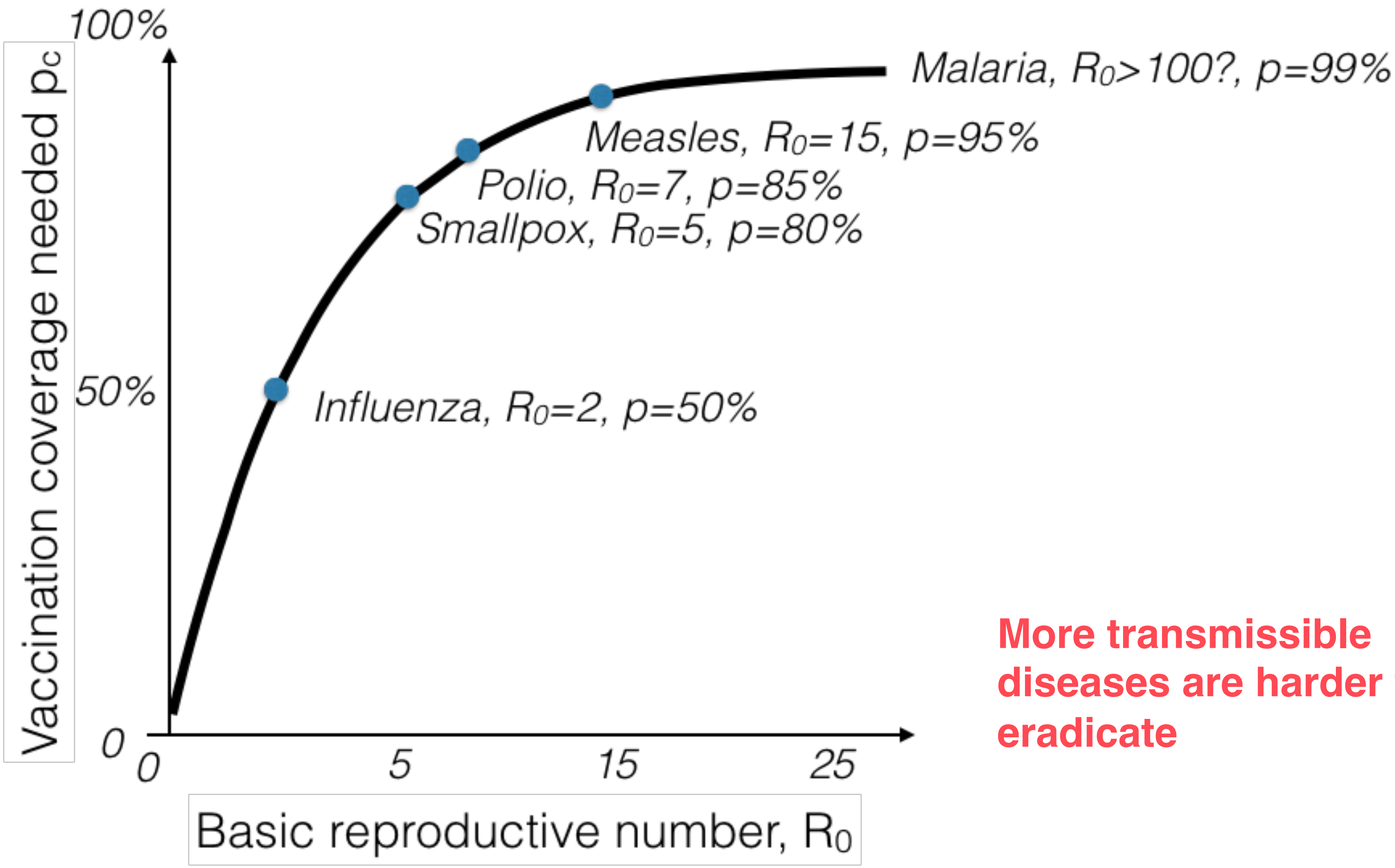
The SIR model : vaccination



$$p_c = 1 - \frac{1}{R_0}$$

The SIR model : eradication

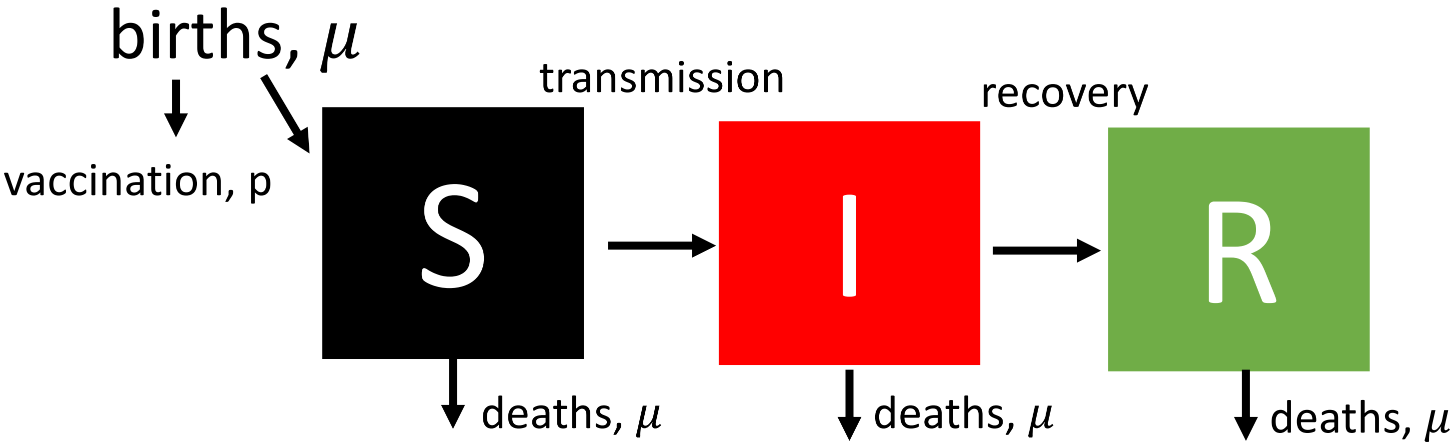
$$p_c = 1 - \frac{1}{R_0}$$



More transmissible diseases are harder to eradicate

The SIR model : extensions

Moving beyond a ‘closed’ population



The SIR model : extensions

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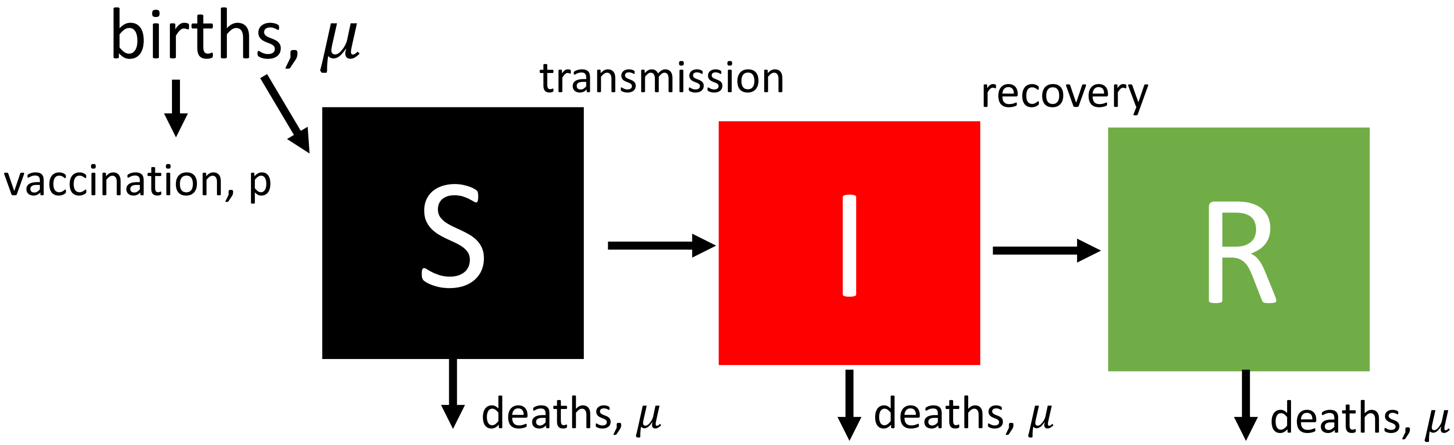

$$\frac{dS(t)}{dt} = \mu(1-p) - \beta S(t)I(t) - \mu S(t)$$

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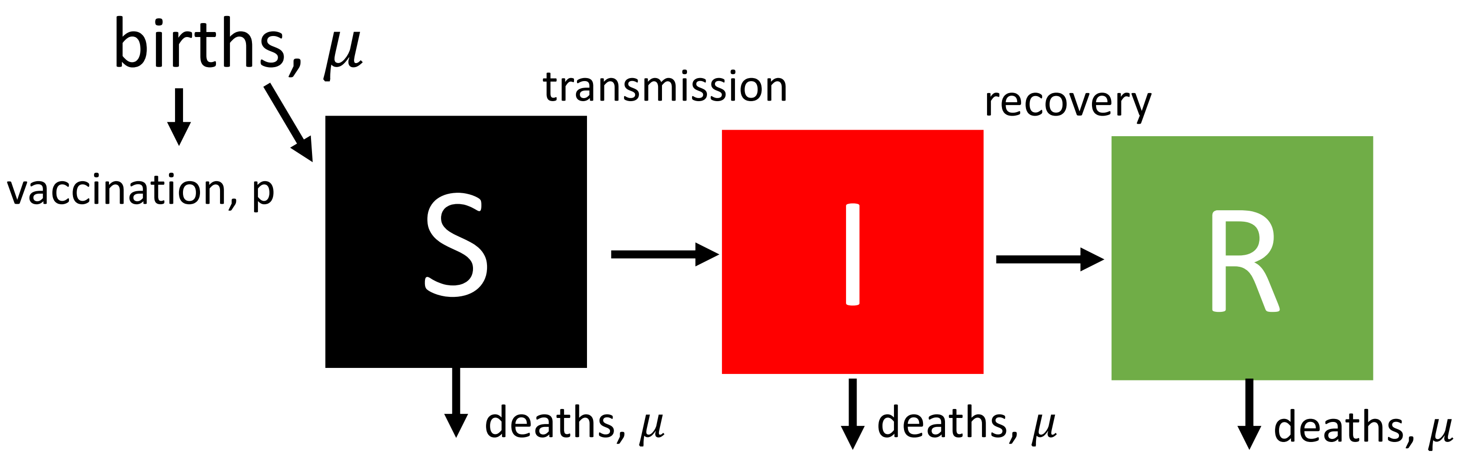
The SIR model : add births

Moving beyond a ‘closed’ population



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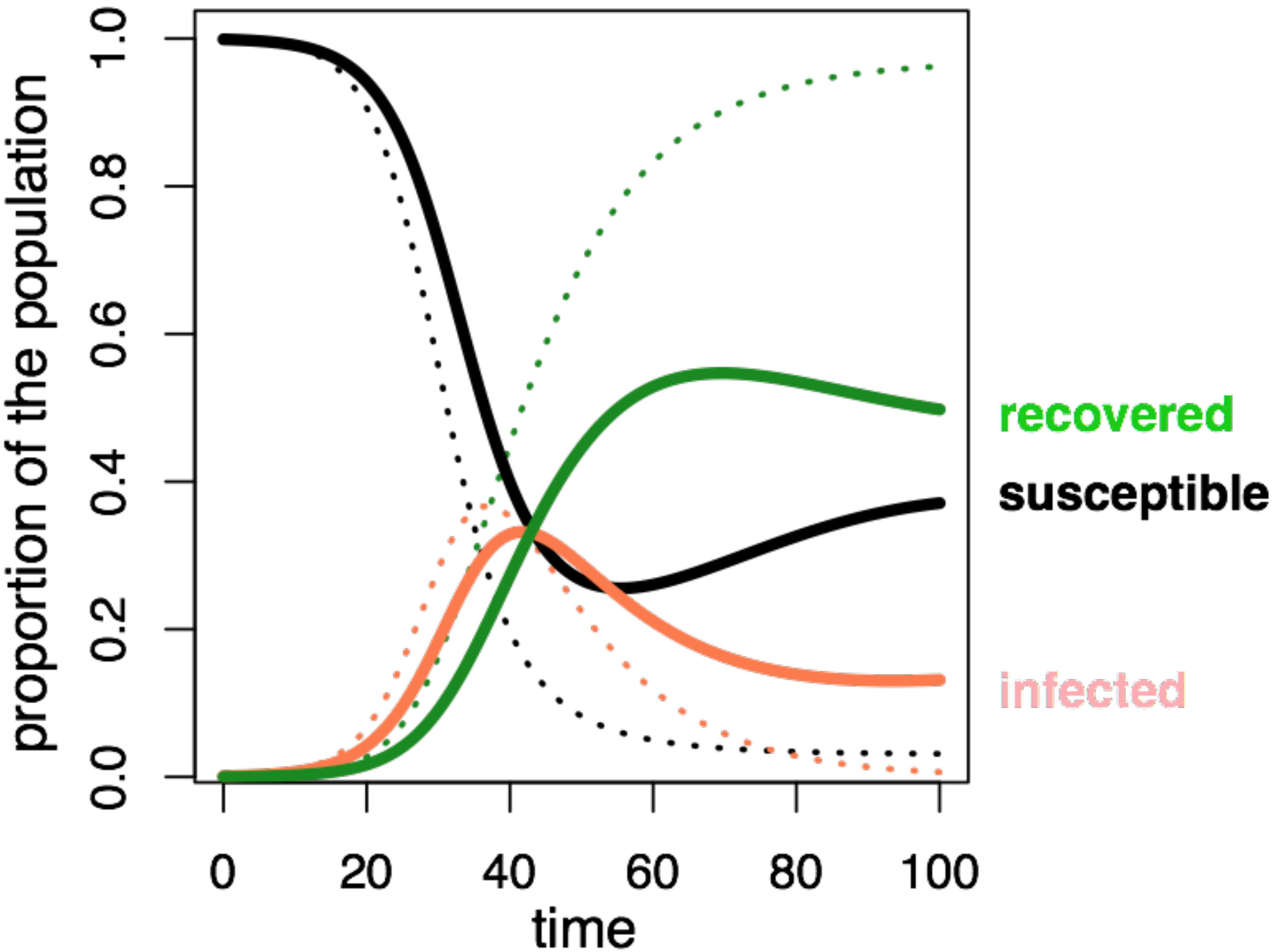
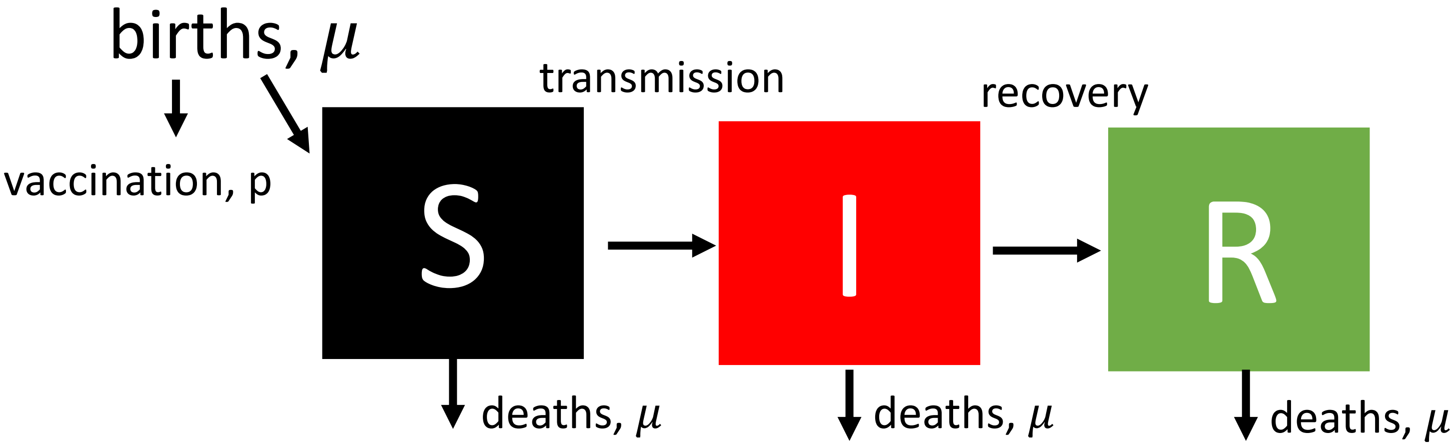
$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I$$

What is likely to be the BIGGEST dynamical difference?

The SIR model : add births

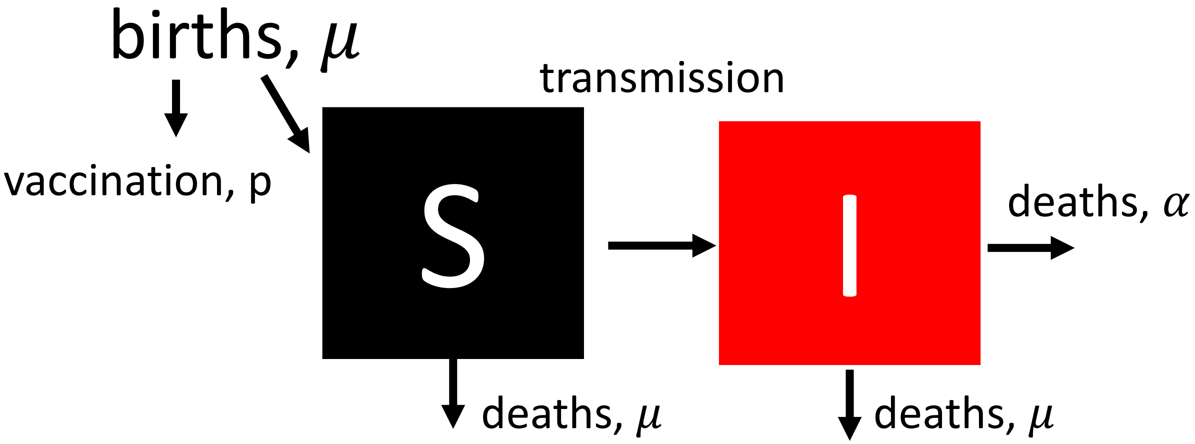


Beyond the SIR model



What do we change if infection is always FATAL?

Beyond the SIR model



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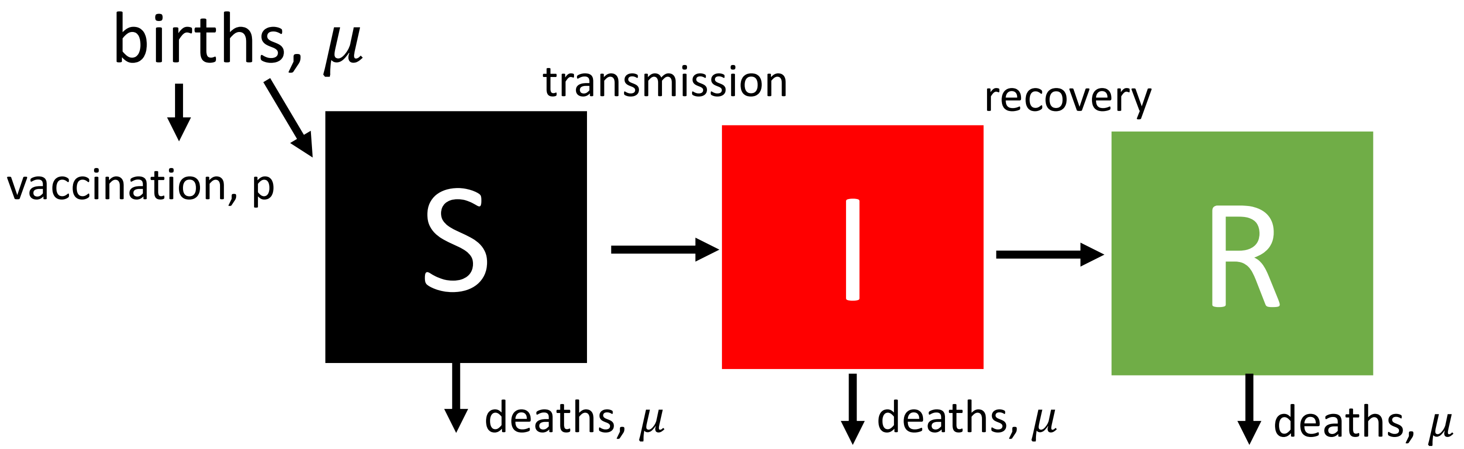
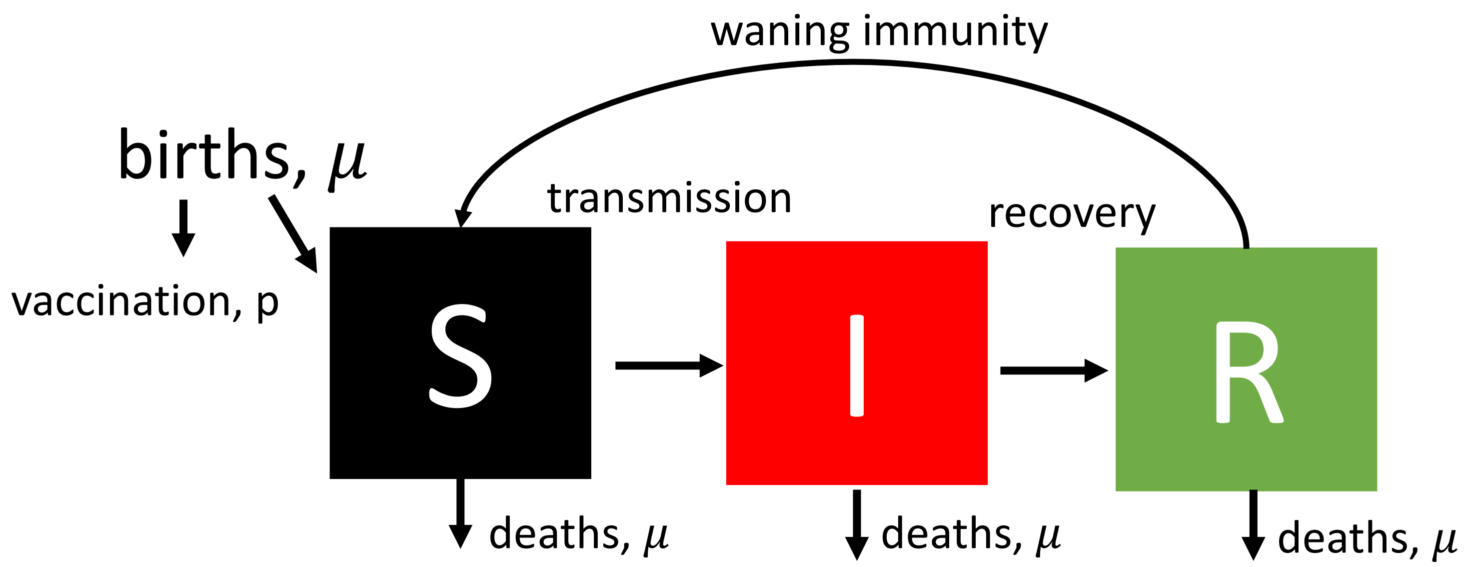


What if immunity wanes?

Beyond the SIR model



What if immunity wanes?

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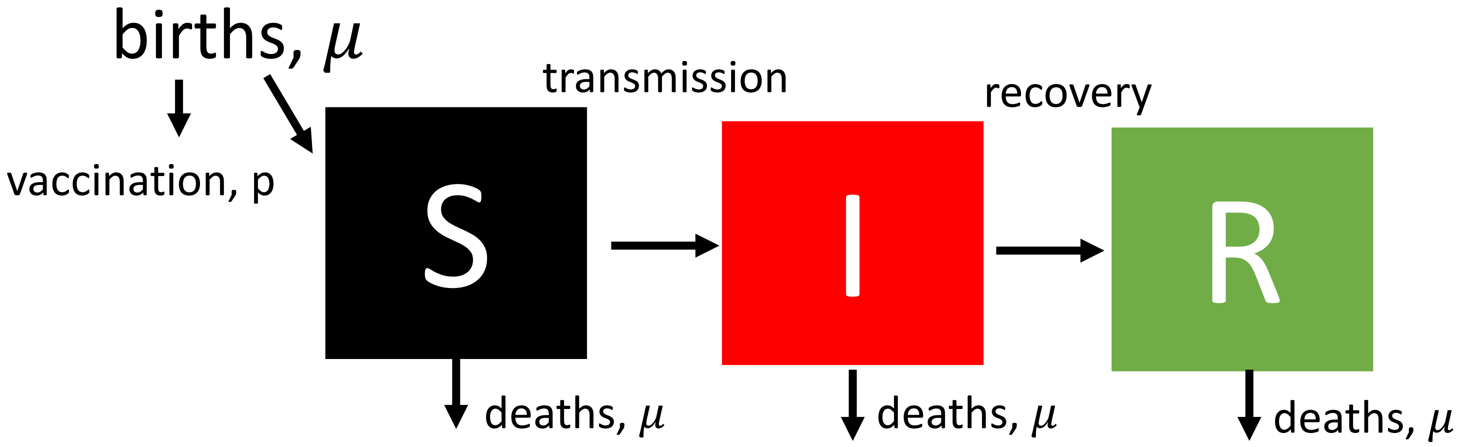
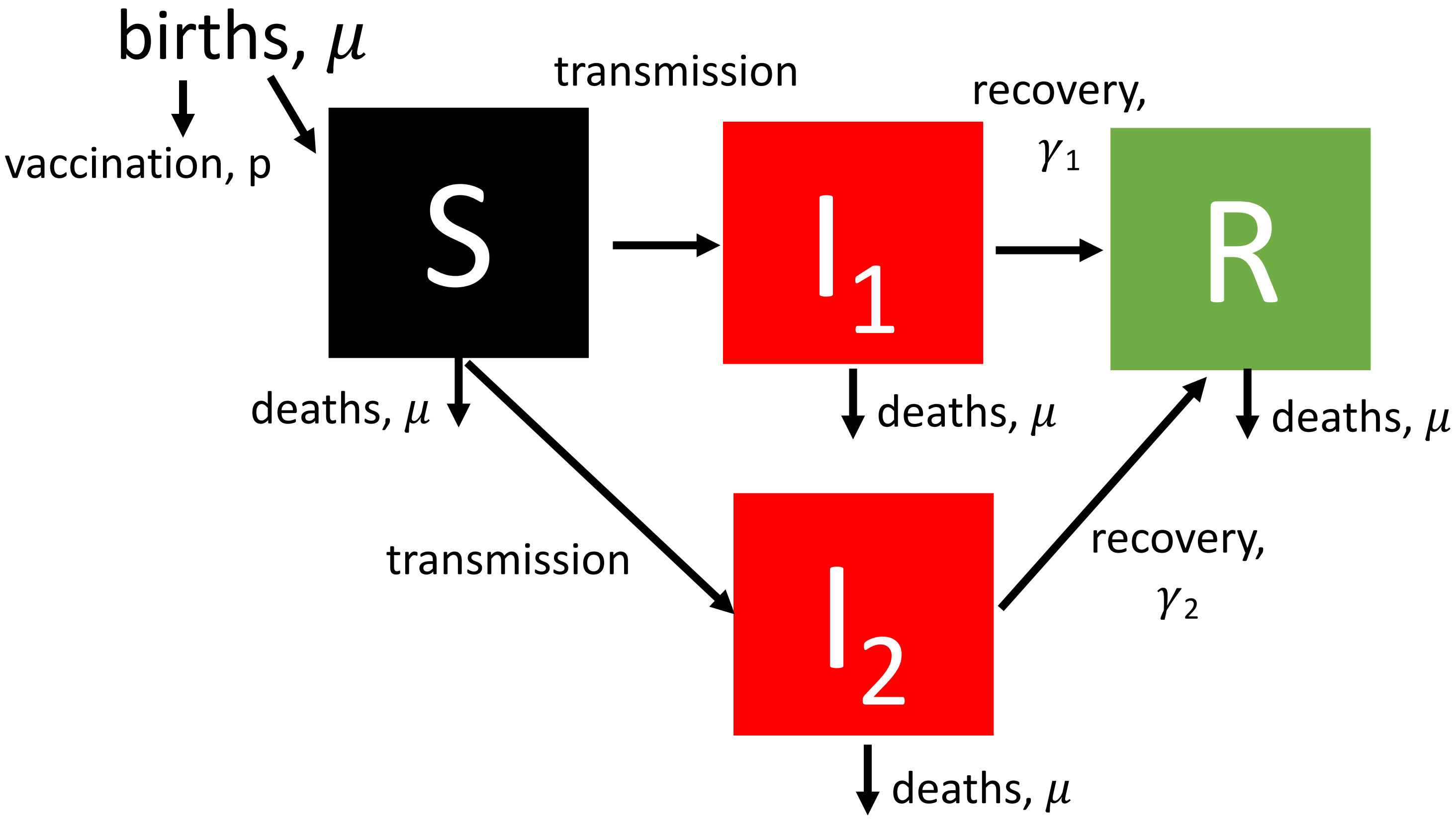


What if people recover at different rates?

Beyond the SIR model



What if people recover at different rates?

Key concepts

- Compartmental/mechanistic/mathematical models

  Modèles en compartiments

- Continuous vs. discrete models

  Modèles en temps continue vs. modèles en temps discrète

- Deterministic vs. stochastic models

  Modèles détérministique vs. stochastique

- Structured models

  Modèles structurés.

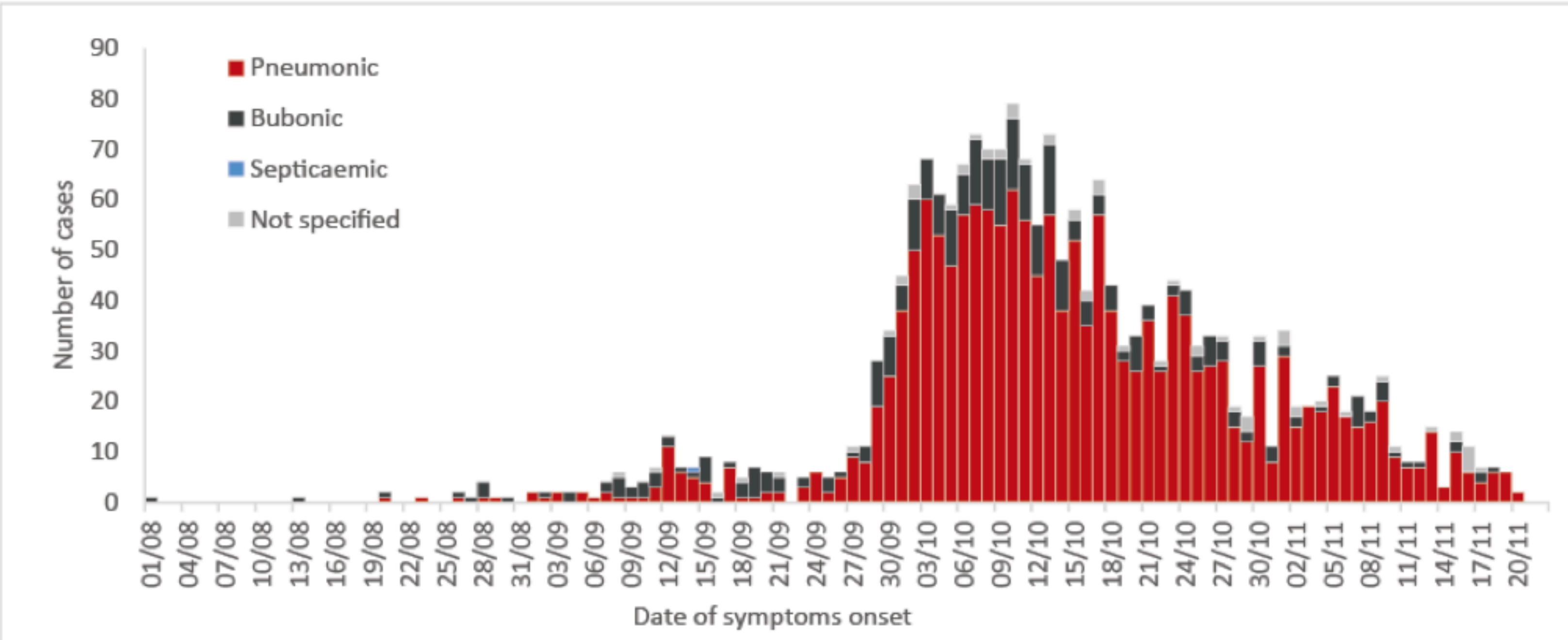
- Two population models

  Modèles des deux populations

- SIR models – and beyond!

  Modèles SIR – et au délà!

Which model?



http://www.who.int/csr/don/27-november-2017-plague-madagascar/en/